Analysis of System Performance

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Chapter 2 – Random Process

Part 2

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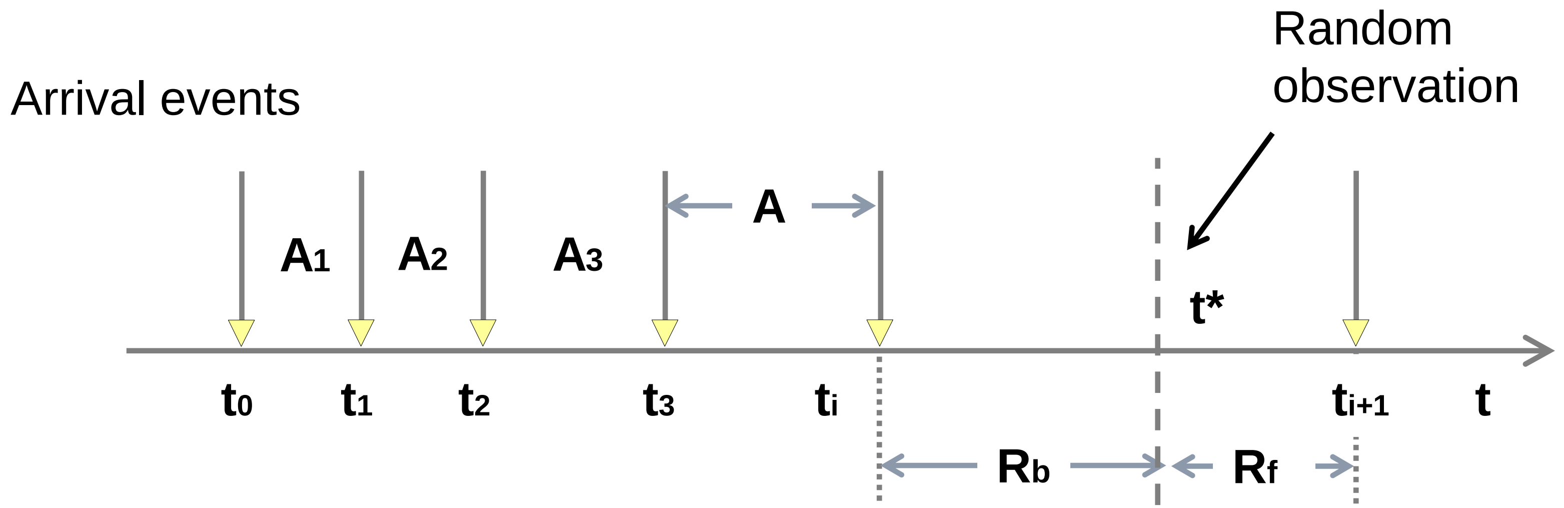
Renewal Process

Process routines in distributed (communication) systems are usually described by arrival processes. Arrival processes are often characterized by renewal processes.

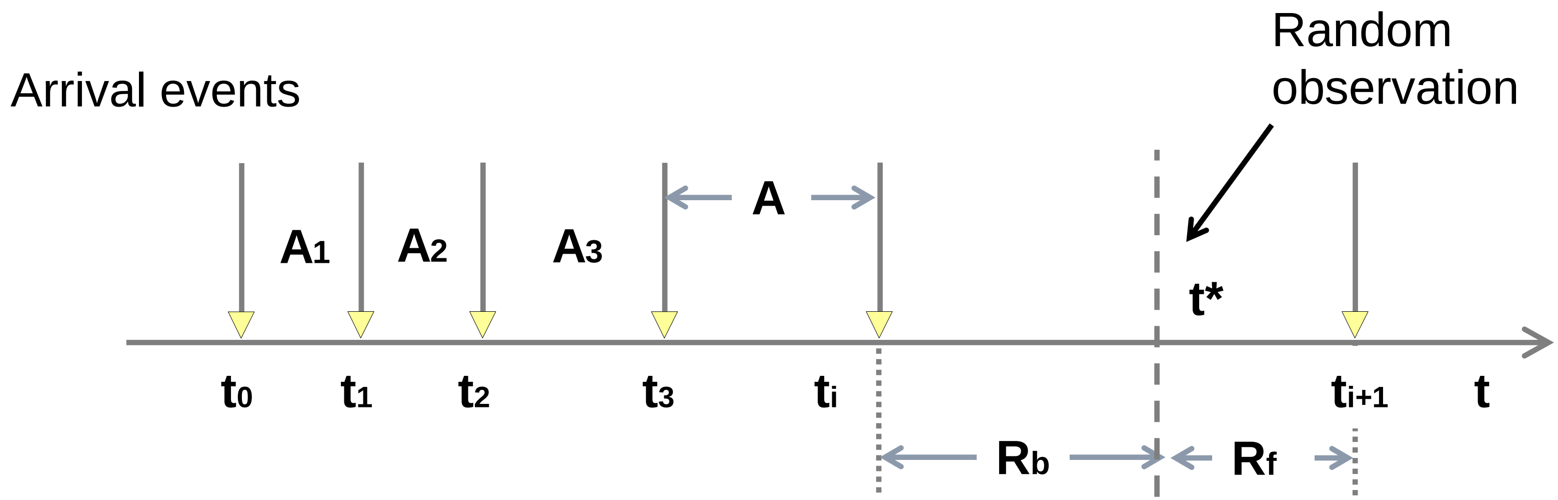


Definition:

A point process is called renewal process if the distances between consecutive events is independent and identically distributed (iid).

$$A_i(t) = A(t), \forall i$$

Recurrence time



Variables:

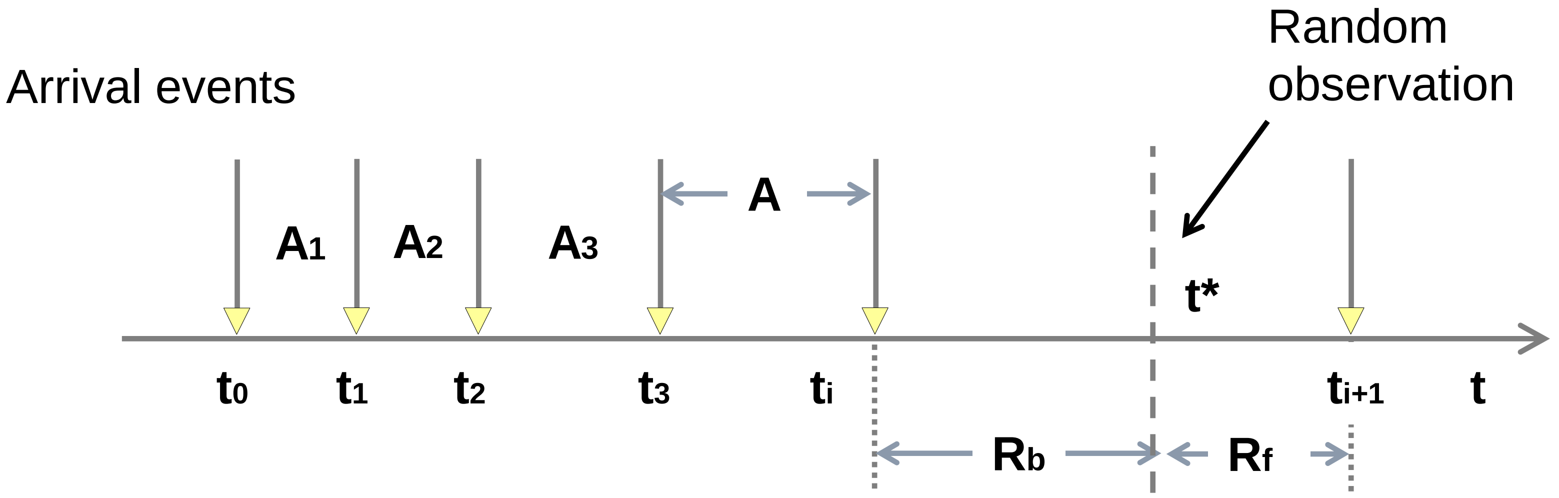
- A : Random variable of the interrarrival time
- A(t) : Distribution function of the RV
- a(t) : Probability density function of the RV
- t* : Random point of observation
- R_f : Forward recurrence time – time intervall between random ovservation time and next event
- R_b : Backward recurrence time – time intervall between previous event and random observation time

Recurrence time - Analysis

- Distribution function:

  The probability density function $r(t)$ of the recurrence time R of a renewal process can be calculated from the distribution function $A(t)$ of the interrarrival time A.

$$r(t) = \frac{1}{E[A]}(1 - A(t)) = \lambda A^C(t) = \lambda \int_{r=t}^{\infty} a(\tau) d\tau$$



Recurrence time - Analysis

Event: Observation point t* falls within an interval with duration $A=\tau$

$a(\tau)$ Probability density of occurrance of an interval with duration τ

$q_\tau = a(\tau)\cdot\tau\cdot n_0$ Probability that the observation point falls within an interval of length τ. The probability is proportional to the interval duration since a longer time interval is observed with a higher probability.

n_0 Constant used to normalise the density function

$$\Rightarrow \int_{r=0}^{\infty} q_\tau d\tau = \int_{r=0}^{\infty} a(\tau)\cdot\tau\cdot n_0 d\tau = n_0 \int_{r=0}^{\infty} a(\tau)\cdot\tau\cdot d\tau = n_0 E[A] = 1$$

$$\Rightarrow n_0 = \frac{1}{E[A]} = \lambda \qquad \Rightarrow q_\tau = \lambda\cdot\tau\cdot a(\tau)$$

Recurrence time - Analysis

Observation point t* lies random distributed within the interval of length $A = \tau$. The conditional probability of the recurrence time is then given by:

$$r(t \mid A = \tau) = \begin{cases} \frac{1}{\tau} & \text{für } t \in (0, \tau) \\ 0 & \text{sonst.} \end{cases}$$

The probability density function of the recurrence time can then be calculated by applying the law of total probability.

$$\Rightarrow \quad r(t) = \int_{\tau=0}^{\infty} r(t \mid A = \tau) \cdot q_t \, d\tau = \int_{\tau=t}^{\infty} \frac{1}{\tau} \cdot \lambda \cdot a(\tau) d\tau = \lambda \Bigg| \int_{\tau=t}^{\infty} a(\tau) d\tau = \lambda A^C(t)$$

q.e.d.

Recurrence time - Analysis

Characteristics:

- Probability density function of the recurrence time $r(t)$ can be calculated if the probability density function of the interarrival time $a(t)$ is known
- Distribution function of the interarrival time $A(t)$ cannot be calculated from the pdf of the recurrence time $r(t)$.

The pdf of the interarrival time $a(t)$ can be calculated if the pdf of the recurrence time and the mean of the interarrival time $E[A]$ are known.

$$r(t) = \frac{1}{E[A]}(1 - A(t)) = \lambda A^C(t) = \lambda \int_{r=t}^{\infty} a(\tau) d\tau$$

Recurrence time - Analysis

- Mean of recurrence time:

$$E[R] = \frac{E[A^2]}{2E[A]} = \frac{c_A^2 + 1}{2} \cdot E[A]$$

⇨ $c_A < 1 \quad : \quad E[R] < E[A]$

⇨ $c_A > 1 \quad : \quad E[R] > E[A]$

The mean of the recurrence time of a renewal process is larger than the mean of the interarrival time, if the variaton coefficient of the interarrival time is larger than one ($c_A > 1$).

⇨ A high variation of the interarrival time leads to large intervals which are likely to be hit by the observer. These intervals contribute more to the recurrence time.

Poisson process

- Definition:

  Poisson process is a renewal process which has a negativ-exponential distributed interarrival time.

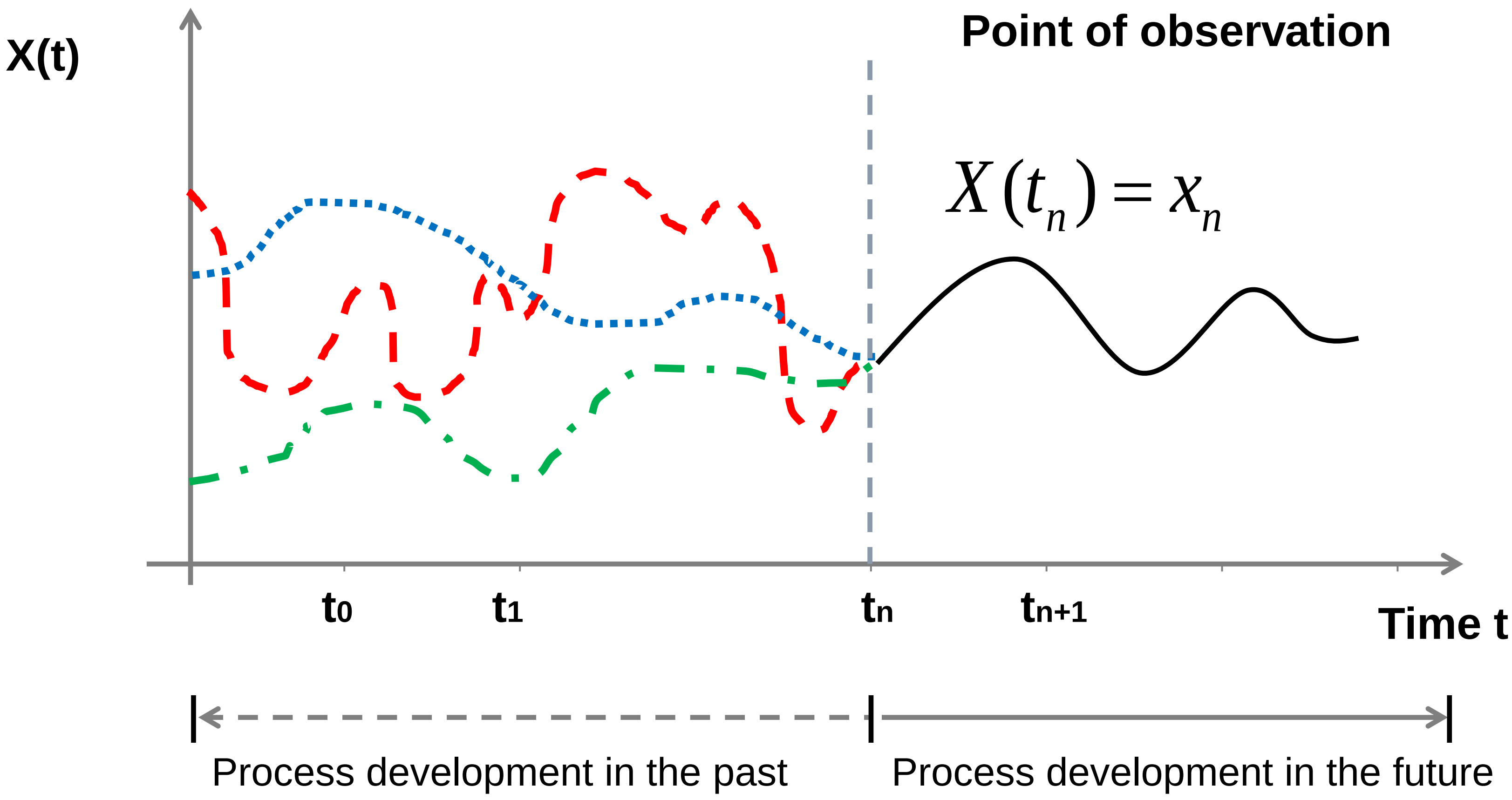
$$A(t) = 1 - e^{-\lambda t}, \qquad a(t) = \lambda e^{-\lambda t}$$

$$\Rightarrow \quad r(t) = \lambda A^C(t) = \lambda(1 - A(t)) = \lambda(1 - 1 + e^{-\lambda t}) = \lambda e^{-\lambda t} = a(t)$$

$$\Rightarrow \quad R(t) = A(t)$$

- The interarrival time and the recurrence time of a poisson process follow the same distribution time.
- The time until the next event, from the perspective of an independent observer, corresponds to the interarrival time. Thus, the process develops independent from its past.
- Poisson process is memoryless (markov property).

Markovian process



Markovian process

This section describes time continuous renewal processes (markovian processes) with discrete states.

- **Transient behavior of markovian processes:**

  The future development of a markovian process **only** depends on its current state and not on its behavior in the past.

$$P\{X(t_{n+1}) = x_{n+1} | X(t_n) = x_n, \dots, X(t_0) = x_0\} =$$
$$P\{X(t_{n+1}) = x_{n+1} | X(t_n) = x_n\}, t_0 < t_1 < \dots < t_n < t_{n+1}.$$

- **Markov chain:**

  A markov chain is a markovian process with finite or countable (discrete) state space.

Markov chains will be discussed in a seperate section

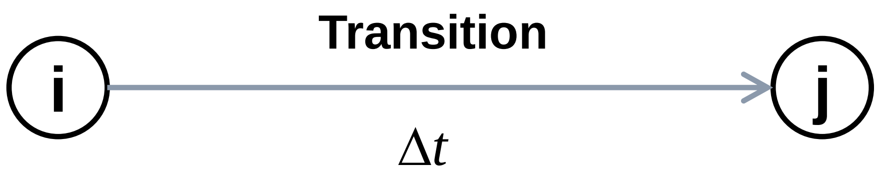
Markovian process

- **Transition probability:**

  Transition probability represents the probility that a process changes from state i at time t_n to state j at time t_{n+1}.

Time t_n $t_{n+1} = t_n + \Delta t$

State



State transition $X(t_n) = i$ → $X(t_{n+1}) = j$

The state changes from i to j within the time interval Δt with the state transition probability:

$$p_{ij}(t_n, t_{n+1}) = P\{X(t_{n+1}) = j \mid X(t_n) = i\}$$

Markovian process

- Homogeneous random process:

  A process is called homogenous if its transition behavior is independent of the oberservation time.

$$\Rightarrow \quad p_{ij}(t_n, t_{n+1}) = p_{ij}(t_n - t_{n+1}) = p_{ij}(\Delta t)$$

Law of total probability:

$$\Rightarrow \quad \sum_j p_{ij}(\Delta t) = 1, \qquad \Delta t \geq 0, \qquad \forall i$$

Markovian process

The transition probability $p_{ij}(\Delta t)$ represents the transition behavior of the process during the time interval with duration Δt. The transition probabilities for every state can be summarized in a transition matrix as follows:

$$\Rightarrow \quad p_{ij}(t_n, t_{n+1}) = p_{ij}(t_n - t_{n+1}) = p_{ij}(\Delta t)$$

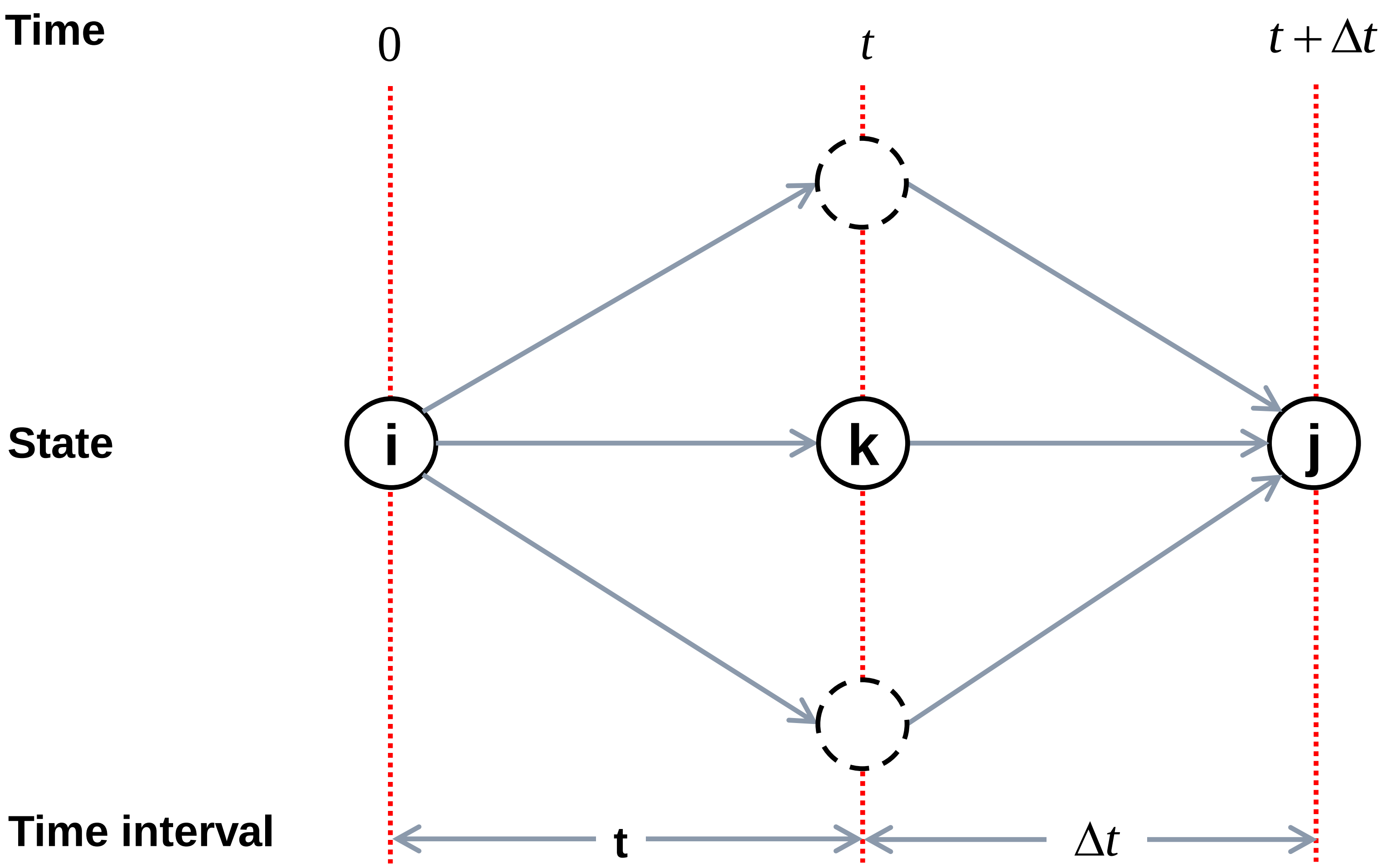
State transition probability

$$\Rightarrow \quad P(\Delta t) = \begin{pmatrix} p_{11}(\Delta t) & p_{12}(\Delta t) & \cdots & p_{1j}(\Delta t) & \cdots \\ p_{21}(\Delta t) & p_{22}(\Delta t) & \cdots & p_{2j}(\Delta t) & \cdots \\ \vdots & \vdots & & \vdots & \\ p_{i1}(\Delta t) & p_{i2}(\Delta t) & \cdots & p_{ij}(\Delta t) & \cdots \\ \vdots & \vdots & & \vdots & \end{pmatrix}$$

State transition matrix

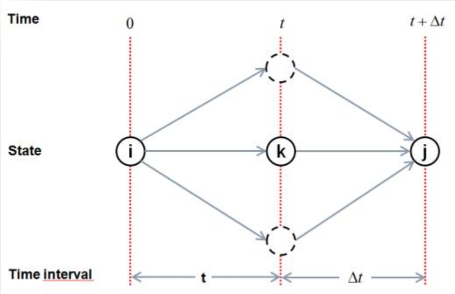
State equations and probabilities

- Transition



State equations and probabilities

Considerations:



- The state of a system changes from one state to another within a certain time interval. During this time interval the system may pass different „intermediate“ states.
- Assume that a system starts in state i at time t1 and reaches state j at time t2. Thus, the system was in an „intermediate“ state k during the time t1 < s < t2.
- The probability that the system changes from state i to state j during time t1 and t2 can be described as follows:

  Probability of changing from state i to ANY state k within t1 and s multiplied by the probability of changing from state k to state j within s and t2.

- Intermediate state k can be any state of the system.
- The product of the state change has to be summarized over all possible „ways“ between state i and state j.

Chapman-Kolmogorov Equation

- Kolmogorov-Forward:

  Assume a process which is in a start state i at time t0 and develops to a state j at time $t + \Delta t$.

  The transition probabilities for the intervals t and delta t are P(t) and P(Δt), respectively. The transition matrix is then given by their product:

$$\Rightarrow \quad P(t + \Delta t) = P(t) \cdot P(\Delta t)$$

$$\Rightarrow \quad p_{ij}(t + \Delta t) = \sum_k p_{ik}(t) \cdot p_{kj}(\Delta t)$$

Chapman-Kolmogorov Equation

$$\Rightarrow \quad p_{ij}(t+\Delta t) = \sum_k p_{ik}(t) \cdot p_{kj}(\Delta t)$$

$$\frac{p_{ij}(t+\Delta t) - p_{ij}(t)}{\Delta t} = \sum_{k \neq j} p_{ik}(t) \cdot \frac{p_{kj}(\Delta t)}{\Delta t} - p_{ij}(t)\frac{1 - p_{jj}(\Delta t)}{\Delta t}$$

with $\Delta t \to 0$

Deviation of transition probabilities $p_{ij}(t)$ at time t

$$\lim_{\Delta t \to 0} \frac{p_{ij}(t+\Delta t) - p_{ij}(t)}{\Delta t} = \frac{d}{dt} p_{ij}(t)$$

Transition probability density for state change k $\rightarrow$ j

$$q_{kj}, \qquad k \neq j$$

Transition probability density for leaving state j

$$\lim_{\Delta t \to 0} \frac{1 - p_{ij}(\Delta t)}{\Delta t} = q_j = \sum_{k \neq j} q_{jk}$$

Chapman-Kolmogorov Equation

- Kolmogorov-Forward:

  With $\Delta t \rightarrow 0$ the state change probability transforms into a state change probability density or state change rate. It describes the possibility that a state changes within an infinitesimal time interval Δt.

Kolmogorov forward equation for transition probabilities ⇨

$$\frac{d}{dt} p_{ij}(t) = \sum_{k \neq j} q_{kj} \cdot p_{ik}(t) - q_j p_{ij}(t)$$

Kolmogorov-Forward

Transition probability density matrix / rate matrix:

$$Q = \begin{pmatrix} q_{00} & q_{01} & \cdots & q_{0j} & \cdots \\ q_{10} & q_{11} & \cdots & q_{1j} & \cdots \\ \vdots & \vdots & & \vdots & \\ q_{j0} & q_{j2} & \cdots & q_{jj} & \cdots \\ \vdots & \vdots & & \vdots & \end{pmatrix}$$

$$\sum_k q_{jk} = 0$$ Probility density for entering and leaving state j

$$q_{jj} = -\sum_{k \neq j} q_{jk} = -q_j$$ Probility density for remaining in state j

$$\frac{dP(t)}{dt} = P(t) \cdot Q$$ Kolmogorov-Forward

Kolmogorov-Forward equation is typically used to analyze the state probabilities of a system.

Kolmogorov-Forward vs. Kolmogorov-Backward

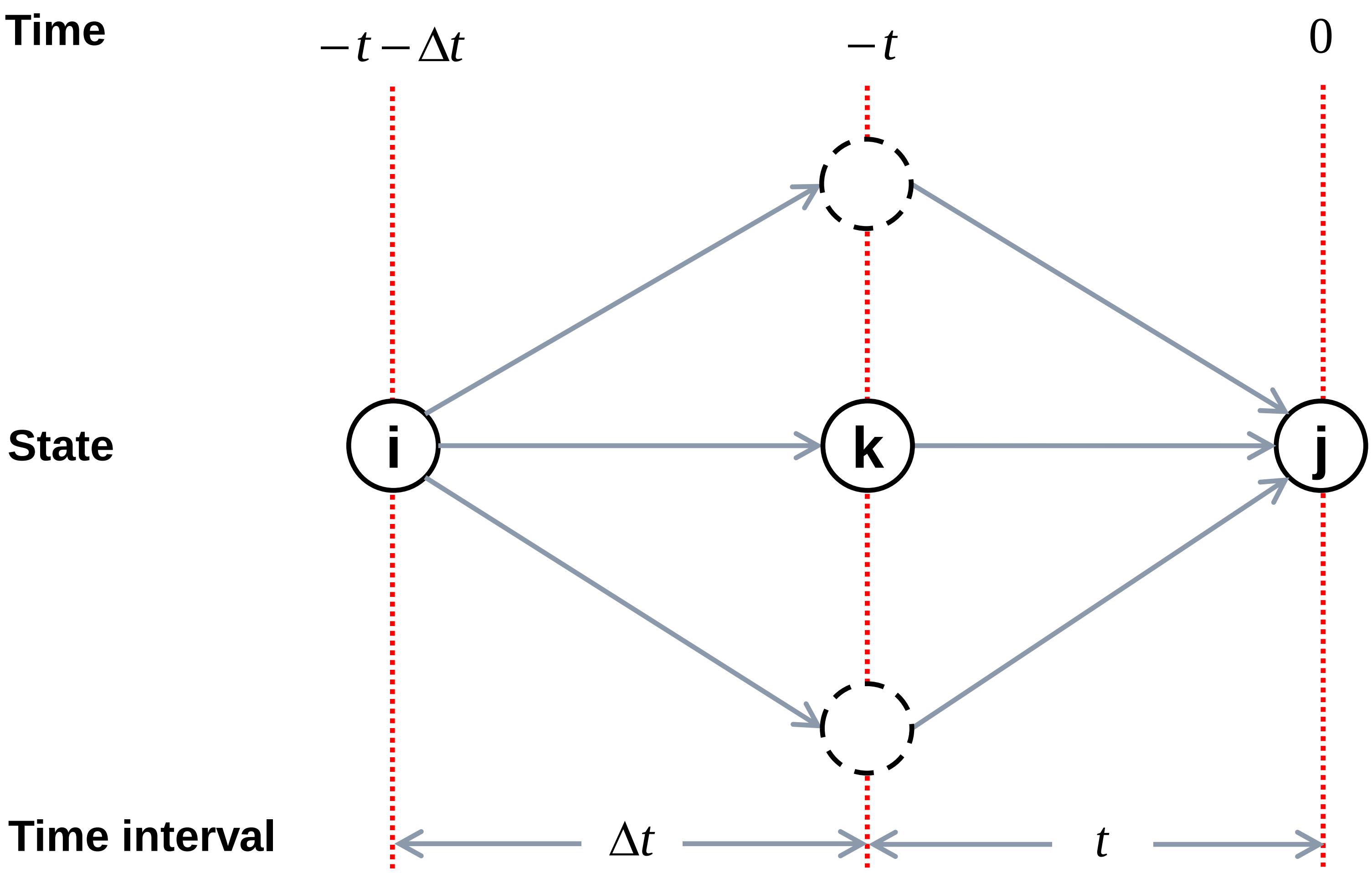
- Kolmogorov-Forward:

  The Kolmogorov-Forward equation is used to evaluate the future development of a process from ist current state.

- Kolmogorov-Backward:

  The Kolmogorov-Backward equation is used to evaluate the development (path) of a process from its current state.

- Transition



Chapman-Kolmogorov Equation

- Kolmogorov-Backward:

  Assume a process which is in state j at the point of observation $t = 0$. The start state at time $-t - \Delta t$ is i. Its development path goes through various intermediate states k.

  The transition probabilities for the intervals t and Δt are $P(t)$ and $P(\Delta t)$, respectively. The transition matrix is then given by their product:

$$\Rightarrow \quad P(t + \Delta t) = P(\Delta t) \cdot P(t)$$

$$\Rightarrow \quad p_{ij}(t + \Delta t) = \sum_k p_{ik}(\Delta t) \cdot p_{kj}(t)$$

Chapman-Kolmogorov Equation

⇒ $$p_{ij}(t+\Delta t) = \sum_k p_{ik}(\Delta t) \cdot p_{kj}(t)$$

$$\frac{p_{ij}(t+\Delta t) - p_{ij}(t)}{\Delta t} = \sum_{k \neq j} \frac{p_{ik}(\Delta t)}{\Delta t} \cdot p_{kj}(t) - p_{ij}(t) \frac{1 - p_{ii}(\Delta t)}{\Delta t}$$

with $\Delta t \to 0$

Deviation of transition probabilities $p_{ij}(t)$ at time t

$$\frac{d}{dt} p_{ij}(t) = \sum_{k \neq i} q_{ik} \cdot p_{kj}(t) - p_{ij}(t) \cdot q_i$$

$q_i \leftrightarrow -q_{ii}$

$$= \sum_k q_{ik} \cdot p_{kj}(t)$$

⇒ $$\frac{dP(t)}{dt} = Q \cdot P(t)$$

Kolmogorov-Backward

Kolmogorov-Backward equation is often applied to evaluate the retention time.

State probabilities

- Kolmogorov-forward equation for state probabilities:

  The equation describes the development of a state process X(t) which is in state j at time t.

$$x(j,t) = P\{X(t) = j\}, \quad j = 0,1,2,\dots$$

⇨ The start state $x(i,0)$ can be derived from $x(j,t)$ by applying the law of total probability.

$$x(j,t) = \sum_i P\{X(t) = j \mid X(0) = i\} \cdot P\{X(0) = i\} = \sum_i x(i,0) \cdot p_{ij}(t)$$

⇨ $$\frac{\partial}{\partial t} x(j,t) = \sum_{k \neq j} q_{kj} \cdot x(k,t) - q_j \cdot x(j,t), \ \forall j$$

Kolmogorov-forward equation for state probabilities

⇨ $$\sum_j x(j,t) = 1$$

Stationary state - system of equation

Steady state:

A state process has reached its steady state if its state probabilities do not change anymore.

$$\Rightarrow \quad \frac{d}{dt} P\{X(t) = j\} = \frac{\partial}{\partial t} x(j,t) = 0$$

Steady state – state probability:

$$\Rightarrow \quad x(j) = \lim_{t \to \infty} P\{X(t) = j\}, \quad \forall j \qquad \sum_j x(j) = 1$$

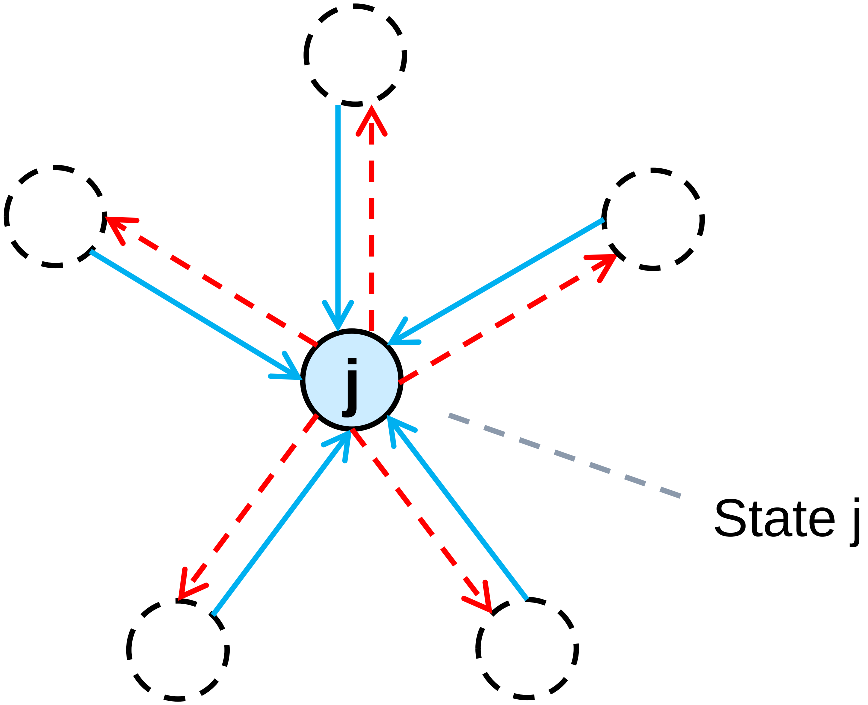
$$\Rightarrow \quad q_j \cdot x(j) = \sum_{k \neq j} q_{kj} \cdot x(k), \quad \forall j$$

Stationary state equation

Probability of leaving state j ($q_j \cdot x(j)$)

Probability of entering state j ($\sum_{k \neq j} q_{kj} \cdot x(k)$)

Steady state



$q_j \cdot x(j)$

Probability density of leaving state j, weighted with state probability $x(j)$.

$\sum_{k \neq j} q_{kj} \cdot x(k)$

Probability density of reaching state j from all other states $k \neq j$, weighted with the corresponding state probability $x(k)$.

Micro state and Macro state

- Micro state:

  A single state that cannot be further be divided is called micro state.

- Macro state:

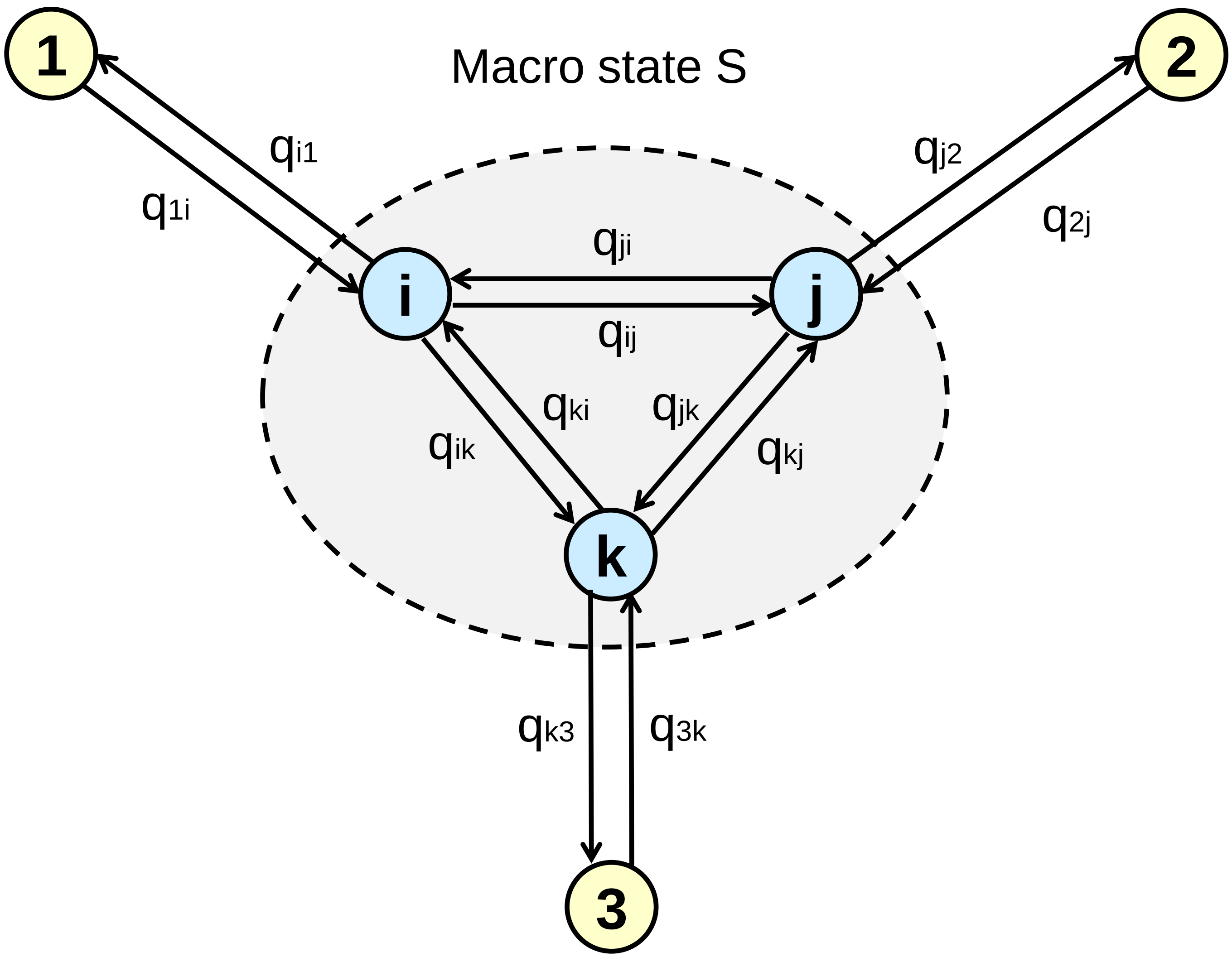
  Multiple micro states can be combined into larger macro states.

⇒ Macro states are typically used to reduce the complexity of a model.

⇒ Macro states should be used if no detailed information about a system is available.

Macro state

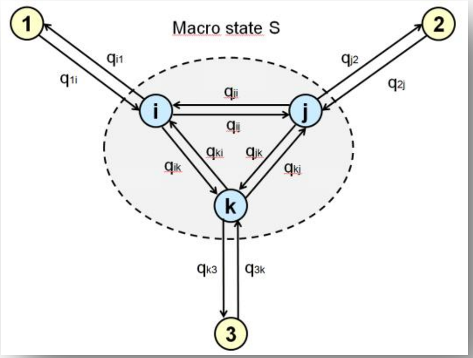
- Equilibrium of macro states



Macro state

Steady state:

The weighted probability densities for entering and leaving the macro state S are equal, if the process is in a steady state.



Sum of stationary system of equilibrium for every micro state within the macro state

$$+\begin{cases} (q_{i1}+q_{ij}+q_{ik})\cdot x(i) = q_{1i}\cdot x(1) + q_{ji}\cdot x(j) + q_{ki}\cdot x(k) \\ (q_{j2}+q_{jk}+q_{ji})\cdot x(j) = q_{2i}\cdot x(2) + q_{kj}\cdot x(k) + q_{ij}\cdot x(i) \\ (q_{k3}+q_{ki}+q_{kj})\cdot x(k) = q_{3k}\cdot x(3) + q_{ik}\cdot x(i) + q_{jk}\cdot x(j) \end{cases}$$

$$q_{i1}\cdot x(i) + q_{j2}\cdot x(j) + q_{k3}\cdot x(k) = q_{1i}\cdot x(1) + q_{2j}\cdot x(2) + q_{3k}\cdot x(3)$$

Weighted probability density of leaving macro state S.

Weighted probability density of entering macro state S.

Macro state

- General state equation of a macro state:

$$\sum_{\substack{j \in S \\ u \notin S}} q_{ju} x(j) = \sum_{\substack{j \notin S \\ u \in S}} q_{uj} x(u)$$

Weighted probability density of leaving macro state S.

Weighted probability density of entering macro state S.

⇨ The equation only contains the state probability of the micro states.

⇨ The state probability of the macro state cannot be calculated from the formulas.

⇨ The macro state probability is given by the sum of the state probabilities if its micro states: $\sum_{j \in S} x(j)$