Discrete Event Simulation
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Chapter 2 - Statistics

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Topics

- Random Variable
- Probability Space
- Discrete and Continuous RV
- Frequency Probability (Relative Häufigkeit)
- Distribution (discrete)
- Distribution Function (discrete)
- PDF, CDF
- Expectation/Mean, Mode,
- Standard Deviation, Variance, Coefficient of Variation
- p-percentile (quantile), Skewness, Scalability Issues (Addition)
- Covariance, Correlation, Autocorrelation
- Visualization of Correlation
- PP-Plot
- QQ-Plot
- Waiting Queue Examples
Classic definition of probability

The theory of chance consists in reducing all the events of the same kind to a certain number of cases equally possible, that is to say, to such as we may be equally undecided about in regard to their existence, and in determining the number of cases favorable to the event whose probability is sought. The ratio of this number to that of all the cases possible is the measure of this probability, which is thus simply a fraction whose numerator is the number of favorable cases and whose denominator is the number of all the cases possible.

- Pierre-Simon Laplace, A Philosophical Essay on Probabilities
Random Variable

Probability Space (Ereignisraum) \( \Omega = \{ \omega_1, \omega_2, \omega_3, \omega_4, \ldots, \omega_i \} \)
Statistics Fundamentals

- **Discrete Random Variable:**
  - Example: Flipping of a coin
    - $\omega_1=${head-0}, $\omega_2=${tail-1}
    - $X \in \{0, 1\}$
  - Example: Rolling two dice
    - $\omega_1=${2}, $\omega_2=${3}, … , $\omega_{11}=${12}
    - $X \in \{2, 3, 4, ..., 12\}$

- **Continuous Random Variable:**
  - Example: Round Trip Time
    - $T \in \{5\text{ms}, 200\text{ms}\}$
    - $\omega_1=${t<10ms}, $\omega_2=${10ms≤t<20ms}, $\omega_3=${t≥20ms} $\Rightarrow \Omega = \{\omega_1, \omega_2\}$
  - Example: Sensed Interference Level

Discrete or not discrete

Countable

Uncountable
Statistics Fundamentals

- **Frequency Probability** / Law of large numbers (Relative Häufigkeit)

  - Number of random experiments
    - $n$: total number of trials
    - $X_i$: event or characteristic of the outcome
    - $n_i$: number of trials where the event $X_i$ occurred

  \[
  h(X_i) = \frac{n_i}{n}, \quad 0 \leq h(X_i) \leq 1, \quad \sum_i h(X_i) = 1 \\
  P(X_i) = \lim_{n \to \infty} \frac{n_i}{n}, \quad 0 \leq P(X_i) \leq 1, \quad \sum_i P(X_i) = 1
  \]

  - Vollständigkeitsrelation
  - $X_i$ disjoint
Statistics Fundamentals

- Vollständiges Ereignissystem
  \[ P(Y) = \sum_{i=1}^{N} P(X_i) \]

- Verbundereignis
  \[ P(X \cap Y) = P(X,Y) = P(Y,X) \]
  \[ P(X \cup Y) = P(X) + P(Y) - P(X \cap Y) \]

- Bedingte Wahrscheinlichkeit
  \[ P(X \mid Y) = \frac{P(X,Y)}{P(Y)} \quad \Rightarrow \quad P(X \mid Y) \geq P(X,Y) \]

- Statistische Unabhängigkeit
  \[ P(X \mid Y) = P(X) \quad \vee \quad P(X,Y) = P(X)P(Y) \]
Vollständiges Ereignissystem

- \[ P(Y) = \sum_{i=1}^{N} P(X_i, Y) \]

Bayes Function

- \[ P(X_i \mid Y) = \frac{P(Y \mid X_i) \cdot P(X_i)}{P(Y)} = \frac{P(Y \mid X_i) \cdot P(X_i)}{\sum_{k=1}^{N} P(Y \mid X_k) \cdot P(X_k)} \]
Statistics Fundamentals

- **Distribution (Verteilung)**

  X – discrete random variable

  - Function $x(i) = P(X = i)$, $i = 0, 1, 2, \ldots, X_{\text{max}}$ (Distribution)
  - $x(i) \in [0, 1]$

  \[ \sum_{i=0}^{X_{\text{max}}} x(i) = 1 \] (Vollständigkeitsrelation)

- Example:
  Rolling two dice
  - $\omega_1 = \{2\}, \omega_2 = \{3\}, \ldots, \omega_{11} = \{12\}$  => $\Omega = \{\omega_1, \omega_2, \ldots, \omega_{11}\}$
  - $X \in \{2, 3, 4, \ldots, 12\}$
Example: Throwing two dice

Sample Space (Ereignisraum)

(1,1) (1,2) (1,3) (1,4) (1,5) (1,6)
(2,1) (2,2) (2,3) (2,4) (2,5) (2,6)
(3,1) (3,2) (3,3) (3,4) (3,5) (3,6)
(4,1) (4,2) (4,3) (4,4) (4,5) (4,6)
(5,1) (5,2) (5,3) (5,4) (5,5) (5,6)
(6,1) (6,2) (6,3) (6,4) (6,5) (6,6)
Statistics Fundamentals

- Example: Throwing two dice

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Statistics Fundamentals

Distribution

Two Dice - Pips

Probability

0 0.02 0.04 0.06 0.08 0.1 0.12 0.14 0.16 0.18

1 2 3 4 5 6 7 8 9 10 11 12
Palour Game: Die Siedler von Catan

- **Rules:**
  - Players are only allowed to build along borders of a field
  - Players roll two dice
  - If the sum of the dice corresponds to the number of the field, the player gets the resources from this field

- **Question**
  - Where is the best place for a building?
Are all fields reached with the same probability?
Statistics Fundamentals

- Probability Mass Function (Verteilung)
  - Discrete random variable $X$
  - $i$ value of the random variable $X$
  - $x(i)$ probability that the outcome of random variable $X$ is $i$

  - $x(i) = P\{X = i\}, \quad i = 0, 1, ..., X_{\text{max}}$ (Distribution)

  - $\sum_{i=0}^{X_{\text{max}}} x(i) = 1$ (Vollständigkeitsrelation)
Statistics Fundamentals

- **Cumulative Distribution Function** (Verteilungsfunktion)

\[ X(t) = P\{X \leq t\} \]

- \( t_1 < t_2 \quad \Rightarrow \quad X(t_1) \leq X(t_2) \) (monotony)

- \( t_1 < t_2 \quad \Rightarrow \quad P\{t_1 < X \leq t_2\} = X(t_2) - X(t_1) \)

- \( X(-\infty) = 0 \quad \land \quad X(\infty) = 1 \)

- \( X^c(t) = 1 - X(t) = P\{X > t\} \)

Cumulative Distribution Function
Difference between probability mass function and cumulative distribution function

Probability Mass Function (Verteilung)

Cumulative Distribution Function (Verteilungsfunktion)
Continuous random variable

- Probability Density Function (Verteilungsdichtefunktion)

\[ x(t) = \frac{d}{dt} X(t) \]

- Cumulative Density Function

\[ X(t) = \int_{-\infty}^{t} x(t) dt \]
Statistics Fundamentals

- **Expectation** (Erwartungswert)
  - $X$: Probability density function
  - $g(x)$: Function of random variable $X$
  
  \[ E[g(X)] = \int_{-\infty}^{\infty} g(t) \cdot x(t) \, dt \]

- **Mean** (Mittelwert einer Zufallsvariablen)
  
  \[ m_1 = E[X] = \int_{-\infty}^{\infty} t \cdot x(t) \, dt \]

- **Mode** (Outcome of the random variable with the highest probability)
  
  \[ c = \text{Max}(x(t)) \]
Statistics Fundamentals

- Gewöhnliche Momente einer Zufallsvariablen
  \[ g(X) = X^k \quad \rightarrow \quad m_k = E[X^k] = \int_{-\infty}^{\infty} t^k \cdot x(t) dt, \quad k = 0, 1, 2, \ldots \]

- Central moment (Zentrales Moment)
  - Variation of the random variable in respect to its mean

  \[ g(X) = (X - m_1)^k \]

  \[ \mu_k = E[(X - m_1)^k] = \int_{-\infty}^{\infty} (t - m_1)^k \cdot x(t) dt, \quad k = 0, 1, 2, \ldots \]

  - Special Case (k=2):

  \[ \mu_2 = E[(X - m_1)^2] = VAR[X] \]
Statistics Fundamentals

- **Standard deviation** (Standardabweichung)
  \[ \sigma_X = \sqrt{\text{VAR}[X]} \]

- **Coefficient of variation** (Variationskoeffizient)
  \[ c_X = \frac{\sigma_X}{E[X]}, \quad E[X] > 0 \]

  - The coefficient of variation is a normalized measure of dispersion of a probability distribution.
  - It is a dimensionless number which does not require knowledge of the mean of the distribution in order to describe the distribution.

Picture taken from Wikipedia
Statistics Fundamentals

- **p-percentile** $t_p$ (p-Quantil)
  
  A percentile is the value of a variable below which a certain percent of observations fall

  - VDF $F : R \rightarrow (0,1)$  
    (bijective)
  
  - $F(x) = P(X < x) = p$
  
  - $F^{-1}(x) = \inf\{x \in R : p \leq F(x)\}$

  - Special Case:
    - Median 0.5-percentile
    - Upper percentile 0.75-percentile
    - Lower percentile 0.25-percentile

  - Typical Use Case:
    - QoS in networks
      (e.g. 99.9%-percentile of the delay)

  - **Cumulative Density Function**
    - 90% of customers are waiting less than 5 minutes

  - 90%-Quantil

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<tr>
<th>Time in s</th>
<th>P(X≤t)</th>
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<tbody>
<tr>
<td>0</td>
<td>0.0</td>
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<tr>
<td>100</td>
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<td>0.2</td>
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<td>600</td>
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<td>700</td>
<td>0.7</td>
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<tr>
<td>900</td>
<td>0.9</td>
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<tr>
<td>1000</td>
<td>1.0</td>
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Skewness (Schiefe)

Skewness describes the asymmetry of a distribution

- $v < 0$: The left tail of the distribution is longer (linksschiefe)
  $\Rightarrow$ Mass is concentrated in the right
- $v > 0$: The right tail of the distribution is longer (rechtsschiefe)
  $\Rightarrow$ Mass is concentrated in the left

\[
\nu_X = E \left[ \left( \frac{X - \mu}{\sigma} \right)^3 \right] = \frac{\mu_3}{\sigma^3}
\]
Scalability Issues

- Multiplication of a random variable $X$ with a scalar $s$
  
  \[ Y = s \cdot X \]

  \[ E[Y] = s \cdot E[X] \]

  \[ \text{VAR}[Y] = s^2 \cdot \text{VAR}[X] \]

- Addition of two random variables $X$ and $Y$

  \[ Z = X + Y \]

  \[ E[Z] = E[X] + E[Y] \]

  \[ \text{VAR}[Z] = \text{VAR}[X] + \text{VAR}[Y] \] (only if $A$ and $B$ independent)
Statistics Fundamentals

- **Covariance**

  Covariance is a measure which describes how two variables change together.

  \[
  \text{Cov}(X,Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X] \cdot E[Y]
  \]

  - **Special Case:** \( \text{Cov}(X, X) = \text{VAR}[X] \)
  
  - **Other Characteristics:**
    
    - \( \text{Cov}(X, a) = 0 \)
    
    - \( \text{Cov}(X, Y) = \text{Cov}(Y, X) \)
    
    - \( \text{Cov}(aX, bY) = ab \text{Cov}(X, Y) \)
    
    - \( \text{Cov}(X + a, Y + b) = \text{Cov}(X, Y) \)
Statistics Fundamentals

- **Correlation function**
  Correlation function describes how two random variables tend to deviate from their expectation

\[
Cor(X, Y) = \frac{Cov(X, Y)}{\sqrt{VAR(X) \cdot VAR(Y)}}
\]

- **Characteristics:**
  - \( Y = X \) \( \Rightarrow \) \( Cor(X, Y) = 1 \) (Maximum positive)
  - \( Y = -X \) \( \Rightarrow \) \( Cor(X, Y) = -1 \) (Maximum negative)
  - \( Cor(X, Y) > 0 \) Both random variables tend to have either high or low values (difference to their expectation)
  - \( Cor(X, Y) < 0 \) The random variables differ from each other such that one has high values while the other has low values and vice versa (difference to their expectation)
Statistics Fundamentals

- **Autocorrelation** (LK 4.9)
  - Autocorrelation is the cross-correlation of a signal with itself. In the context of statistics it represents a metric for the similarity between observations of a stochastic process. From a mathematical point of view, autocorrelation can be regarded as a tool for finding repeating patterns of a stochastic process.

  **Definition:**
  - Correlation of two samples with distance $k$ from a stochastic process $X$ is given by:

    $\text{Cor}(X,Y) \quad \text{with} \quad Y_i = X_{i+j}$

  **Use case:**
  - Test of random number generators
  - Evaluation of simulation results (c.f. Batch-Means)
Example:

Random

Autocorrelation Lag 4
Visualization of Correlation

Example: Two random variables X and Y are plotted against each other.

Picture taken from Wikipedia
Impact of correlation (1/2)

Example: Random Waypoint mobility model

Algorithm

- Node selects a uniform distributed movement speed
- Node moves towards the new position
- Node waits a certain amount of time
- Node chooses a random position
- Node selects a new destination
- Mobile Node
- Scenario Boundary
Impact of correlation (2/2)

Example: Random Waypoint mobility model

- Uncorrelated next position selection
- Correlated next position selection
Statistics Fundamentals

- Mobility Example

![Mobility Example Image]
Statistics Fundamentals

- Visual comparison of different distributions
  - Quantile-Quantile Plot
  - Probability-Probability Plot
Quantile-Quantile plots (QQ plots)

- Usage: Compare two distributions against each other
  - Usually: Measurement distribution vs. theoretical distribution – do the measurements fit an assumed underlying theoretical model?
  - Also possible: Measurement distribution vs. other measurement distribution – are the two measurement runs really from the same population, or is there variation between the two?

- How it works:
  - Determine 1% quantile, 2% quantile, ..., 100% quantile for distributions
  - Plot 1% quantile vs. 1% quantile, 2% quantile vs 2% quantile, etc.
  - Not restricted to percentiles – usually, each of the $n$ data points from the measurement is taken as its own $1/n$ quantile

- How to read:
  - If everything is located along the line $x=y$ then the two distributions are very similar
  - QQ plots amplify discrepancies near the “tail” of the distributions

- Warning about scales:
  - Plot program often automatically assign X and Y different scales
  - Straight line indicates: choice of distribution OK, but parameters don’t fit
Statistics Fundamentals

- QQ Plot

![QQ Plot Diagram]

- Distribution functions:
  - $\tilde{F}_n(x)$
  - $\hat{F}(x)$

- Q–Q plot
  - $x^M$ vs. $x^S$
QQ Plot

Distribution functions

\[ \hat{F}_n(x) \quad \hat{F}(x) \]

\[ x \]

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \]

\[ 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1.0 \]

\[ \hat{F}_n(x) \]

\[ \hat{F}(x) \]

Q–Q plot

\[ x^M \]

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \]

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \]

\[ x^S \]

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \]
Probability-Probability plots (PP plots)

- Very similar to QQ plot
- QQ plot is more common, though

- Difference to QQ plot:
  - QQ plot compares [quantiles of] two distributions: 1% quantile vs. 1% quantile, etc.
    - Graphically: the y axes of the cumulative density distribution functions are plotted against each other
  - PP plot compares probabilities of two distributions
    - Graphically: the y axes of the probability density functions are plotted against each other

- How to read:
  - Basically the same as QQ plot
  - PP plots highlight differences near the centers of the distributions (whereas QQ plots highlight differences near the ends of the distributions)
Statistics Fundamentals

- PP Plot

The difference between the “middles” of \( \hat{F}(x) \) and \( \hat{F}_n(x) \) amplified by the \( P-P \) plot.
The difference between the right tails of $\hat{F}(x)$ and $\hat{F}_n(x)$ amplified by the Q–Q plot.
Monty Hall Problem – (also known as the goat problem)
  - American game show „Let’s make a deal“ adopted in Germany „Geh auf Ganze“
Game rules:

- Behind one door is a price
- Behind the other doors is the goat / Zonk (It is assumed that the candidate is not interested in neither the goat nor the Zonk)
- Candidate may choose one door
- Game master will open one door after the decision of the candidate and will offer the candidate the choice to choose a different door.
Statistics Fundamentals

- **Definition: RV Z=i: „Zonk/Goat is behind door i“**
  - \( P(Z=i)=1/n \) (Laplace)

- **Definition: RV C=i: „Candidate has chosen door i“**
  - \( P(C=i)=1/n \) (Laplace)

- **Z an C are independent**
  - \( P(Z=i \land C=i)=P(Z=i) \cdot P(C=i) \)

- **Definition: ZV O=i: „Door i was opened“**
  - \( P(O=i|Z=i)=0 \) „The winning door was not opened“
  - \( P(Z=i \land O=i) = P(O=i|Z=i) \cdot P(Z=i) = 0 \) (Bayes)
  - \( P(Z=i \land O\neq i) = P(Z=i) – P(Z=i \land O=i) = P(Z=i) \) (Totale Wahrscheinlichkeit)

- **P(O=i|C=i)=0 „The selected door will NOT be opened“**
  - \( P(C=i \land O=i) = P(O=i|C=i) \cdot P(C=i) = 0 \) (Bayes)
  - \( P(C=i \land O\neq i) = P(C=i) – P(C=i \land O=i) = P(C=i) \) (Totale Wahrscheinlichkeit)
Win probability if the player does not change his selection

\[
P(Z = i \mid C = i \land O \neq i) = \frac{P(Z = i \land (C = i \land O \neq i))}{P(C = i \land O \neq i)} = \frac{P(Z = i \land C = i)}{P(Z = i)} = \frac{1}{n \cdot \frac{1}{n}} = \frac{1}{n}
\]

Win probability if player changes his/her selection

\[
P(Z = i \mid C \neq i \land O \neq i) = \frac{1 - P(Z = i \land (C = i \land O \neq i))}{n - 2} = \frac{1 - \frac{1}{n}}{n - 2} = \frac{n - 1}{n \cdot (n - 2)}
\]
Waiting Queue Theory

- **Arrival process**
  - Rate $\lambda$
  - Inter-arrival times:
    - 3.4
    - 0.6
    - 1.7
    - 1.6
    - 0.7
    - 0.7
    - 1.3

- **Buffer, queue**
  - Random numbers:
    - 1.7
    - 1.0
    - 3.3
    - 4.0
    - 0.7
    - ...

- **Service process**
  - Rate $\mu$
  - Service times:
    - 1.3

Statistics Fundamentals
What are we talking about… and why?

- **Simple queue model:**
  - Customers arrive at random times
  - Execution unit serves customers (random duration)
  - Only one customer at a time; others need to queue

- **Standard example**

- **Give deeper understanding of important aspects, e.g.**
  - Random distributions (input)
  - Measurements, time series (output)
  - …
Statistics Fundamentals

Queuing model: Input and output

- **Input:**
  - (Inter-)arrival times of customers (usually random)
  - Job durations (usually random)

- **Direct output:**
  - Departure times of customers

- **Indirect output:**
  - Inter-arrival times for departure times of customers
  - Queue length
  - Waiting time in the queue
  - Load of service unit (how often idle, how often working)
Little Theorem

Little Theorem
Little Theorem

- \( \lambda \) : average arrival rate
- \( E[X] \) : average number of packets in the system
- \( E[T] \) : average retention time of packets in the system

Arrival Process

\[
\begin{align*}
\bar{T} &= \frac{1}{N} \sum_{i=1}^{N} T_i \approx \frac{1}{N} \int_{0}^{t_o} X(t)dt \\
\bar{X} &= \frac{1}{t_o} \int_{0}^{t_o} X(t)dt \quad \Rightarrow \quad \bar{X} \approx \frac{N}{t_o} E[T] \\
\bar{\lambda} &\approx \frac{N}{t_o} \quad \Rightarrow \quad \bar{\lambda} \cdot \bar{T} \approx \bar{X}
\end{align*}
\]

\[
\begin{align*}
\lambda &= \lim_{t_o \to \infty} \bar{\lambda} = \lim_{t_o \to \infty} \frac{N}{t_o} \\
E[T] &= \lim_{t_o \to \infty} \bar{T} = \lim_{t_o \to \infty} \frac{1}{N} \sum_{i=1}^{N} T_i \\
E[X] &= \lim_{t_o \to \infty} \bar{X} = \lim_{t_o \to \infty} \frac{1}{t_o} \int_{0}^{t_o} X(t)dt
\end{align*}
\]
Kendall Notation

GI$^{[x]}$ / GI / n - S

- Number of Places in the Queue
  - $S = 0$ Loss/Blocking System
  - $S = \infty$ Waiting System
- Number of Servers
- Service Time Distribution
- Batch Arrival Process
- Arrival Process
Statistics Fundamentals

- **Queuing Discipline**
  - FIFO / FCFS: First In First Out / First Come First Served
  - LIFO / LCFS: Last In First Out / Last Come First Served
  - SIRO: Service In Random
  - PNPN: Priority-based Service
  - EDF: Earliest Deadline First

- **Distributions**
  - M: Markovian, Exponential Service Time
  - D: Degenerate Distribution, A deterministic service time
  - E_k: Erlang Distribution, Erlang k distribution
  - GI: General distribution, General independent
  - H_k: Hyper exponential, Hyper k distribution
Statistics Fundamentals

- **System Characteristics**
  - Average customer waiting time
  - Average processing time of a customer
  - Average retention time of a customer
  - Average number of customers in the queue
  - Customer blocking probability
  - Utilization of the system / individual processing units

- **Example**

How to model and evaluate waiting queues in OPNET
Questions:

- How does the number of service units affect the system?

- What impact has a higher variance of the arrival and/or service process on the performance of the system?

- Which system has a higher utilization? One with an unlimited number of waiting slots or one with a limited number?

- Which system has a lower retention time? One with many slow serving units or one with a single but fast serving unit?

- How does the queuing strategy (FIFO, LIFO, EDF) affect the average waiting time and the waiting time distribution?
Exercise

- System A: D / D / 1 - ∞
  - Arrival rate $\lambda = 1 / \text{s}$
  - Service rate $\mu = [1;10] / \text{s}$

- System B: M / M / 1 - ∞
  - Arrival rate $\lambda = 1 / \text{s}$
  - Service rate $\mu = [1;10] / \text{s}$

- System C: M / M / 20 - ∞
  - Arrival rate $\lambda = 10 / \text{s}$
  - Service rate $\mu = 1 / \text{s}$

- System D: M / M / 1 - ∞
  - Arrival rate $\lambda = 10 / \text{s}$
  - Service rate $\mu = 20 / \text{s}$

- What is the maximum (meaningful) utilization of the system?
- Which system performs better?
- What impact does the utilization have on the system?
- Which system performs better?
- Would you prefer a single fast processing unit instead of multiple slow processing units?
Exercise

System E: M / M / 10 - ∞
- Arrival rate λ = 9 / s
- Service rate μ = 1 / s

System F: M / M / 100 - ∞
- Arrival rate λ = 90 / s
- Service rate μ = 1 / s

System G: M / D / 1 - ∞
- Arrival rate λ = 1 / s
- Service rate μ = 1 / 0.7 / s

System H: D / M / 1 - ∞
- Arrival rate λ = 1 / s
- Service rate μ = 1 / 0.7 / s

What is the maximum (meaningful) utilization of the system?

Which system performs better?

Which system performs better?

Which system has a shorter avg waiting time?
Statistics Fundamentals

- **System G: M / D / 1 - \( \infty \)**

\[ \lambda = 1 \]

\[ P(T > 0.7s) = 1 - P(T \leq 0.7s) = 0.55 \]

- **System H: D / M / 1 - \( \infty \)**

\[ \mu = 1 / 0.7 \approx 1.43 \]

\[ P(T < 1s) = 0.75 \]
- **M / M / n - \infty**

  - System of equations

\[
\lambda x(i - 1) = i \mu x(i), \quad i = 1, 2, 3, \ldots, n,
\]

\[
\lambda x(i - 1) = n \mu x(i), \quad i = n + 1, ...
\]

\[
\sum_{i=0}^{\infty} x(i) = 1
\]

Picture taken from Tran-Gia, Analytische Leistungsbewertung verteilter Systeme, p. 98
- M / M / 10 - \infty

State Distribution - M / M / 10 - \infty

Picture taken from Tran-Gia, Analytische Leistungsbewertung verteilter Systeme, p. 99
**Statistics Fundamentals**

- **\( M / M / 10 - \infty \)**
  - The waiting probability decreases with an increasing number of processing units (assuming constant utilization)

![Graph](image.png)

- \( n = \) number of processing units
- \( \rho_W \) - Waiting probability

Picture taken from Tran-Gia, Analytische Leistungsbewertung verteilter Systeme, p. 100