Discrete Event Simulation

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Chapter 2 - Statistics

Dr. Alexander Klein
Stephan Günther
Prof. Dr.-Ing. Georg Carle

Chair for Network Architectures and Services
Department of Computer Science
Technische Universität München
http://www.net.in.tum.de

Topics

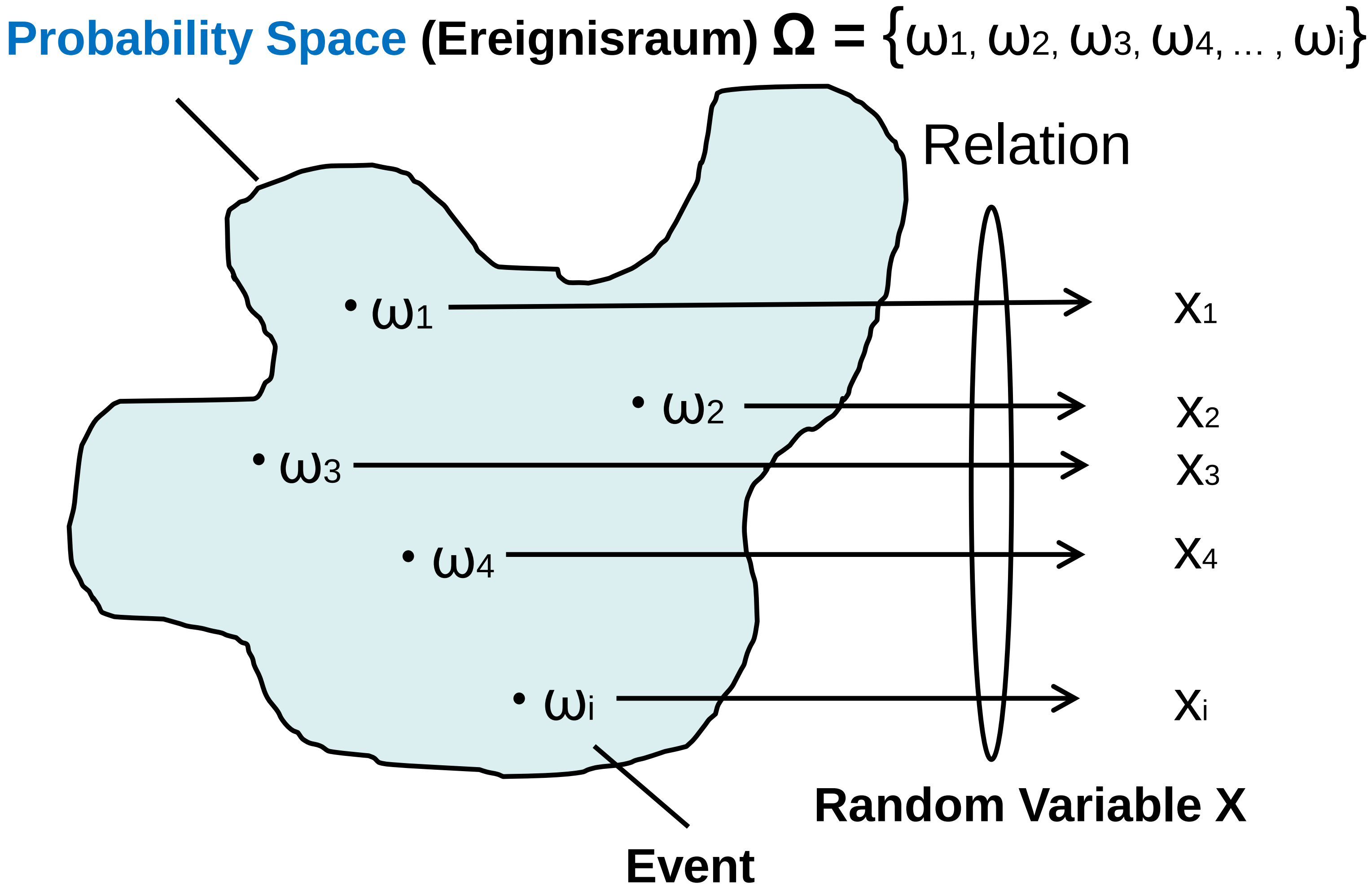
- Random Variable
- Probability Space
- Discrete and Continuous RV
- Frequency Probability(Relative Häufigkeit)
- Distribution(discrete)
- Distribution Function(discrete)
- PDF,CDF
- Expectation/Mean, Mode,
- Standard Deviation, Variance, Coefficient of Variation
- p-percentile(quantile), Skewness, Scalability Issues(Addition)
- Covariance, Correlation, Autocorrelation
- Visualization of Correlation
- PP-Plot
- QQ-Plot
- Waiting Queue Examples

Classic definition of probability

The theory of chance consists in reducing all the events of the same kind to a certain number of cases equally possible, that is to say, to such as we may be equally undecided about in regard to their existence, and in determining the number of cases favorable to the event whose probability is sought. The ratio of this number to that of all the cases possible is the measure of this probability, which is thus simply a fraction whose numerator is the number of favorable cases and whose denominator is the number of all the cases possible.

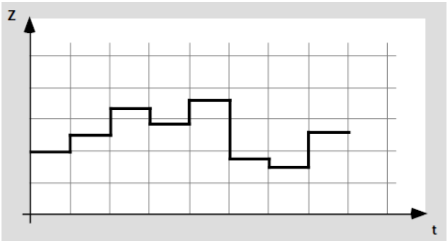
- Pierre-Simon Laplace, A Philosophical Essay on Probabilities

- Random Variable



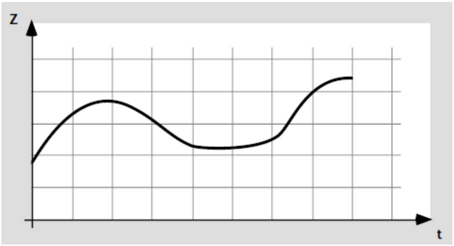
Statistics Fundamentals

- **Discrete Random Variable:**  **Countable**
  - Example: Flipping of a coin
    - ω_1={head-0}, ω_2={tail-1} => $\Omega = \{\omega_1, \omega_2\}$
    - $X \in \{0, 1\}$



  - Example: Rolling two dice 
    - ω_1={2}, ω_2={3}, … , ω_{11}={12} => $\Omega = \{\omega_1, \omega_2, \dots, \omega_{11}\}$
    - $X \in \{2, 3, 4, \dots, 12\}$

- **Continuous Random Variable:** **Uncountable**
  - Example: Round Trip Time
    - $T \in \{5ms, 200ms\}$
    - $\omega_1=\{t<10ms\}$, $\omega_2=\{10ms \le t<20ms\}$, $\omega_3=\{t \ge 20ms\}$ => $\Omega = \{\omega_1, \omega_2\}$
  - Example: Sensed Interference Level



Discrete or not discrete

Statistics Fundamentals

- **Frequency Probability** / Law of large numbers (Relative Häufigkeit)
  - Number of random experiments
    - n total number of trials
    - X_i event or characteristic of the outcome
    - n_i number of trials where the event X_i occurred

$$h(X_i) = \frac{n_i}{n} \qquad 0 \le h(X_i) \le 1 \qquad \sum_i h(X_i) = 1$$

Vollständigkeits-relation

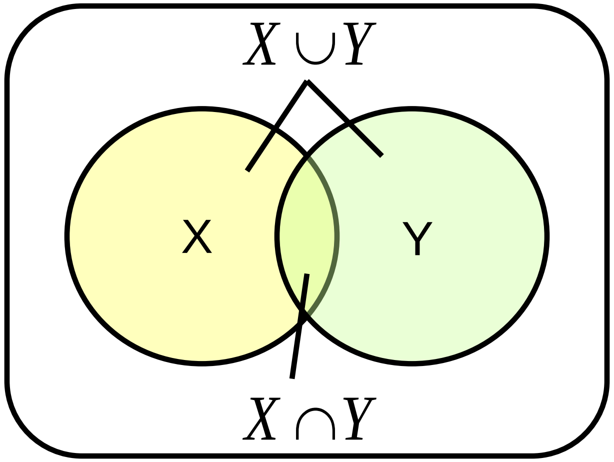
$$P(X_i) = \lim_{n \to \infty} \frac{n_i}{n} \qquad 0 \le P(X_i) \le 1 \qquad \sum_i P(X_i) = 1$$

X_i disjoint

Statistics Fundamentals

- **Vollständiges Ereignissystem**

$$P(Y) = \sum_{i=1}^{N} P(X_i)$$



- **Verbundereignis**

$$P(X \cap Y) = P(X,Y) = P(Y,X)$$

$$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$$

- **Bedingte Wahrscheinlichkeit**

$$P(X \mid Y) = \frac{P(X,Y)}{P(Y)} \Rightarrow P(X \mid Y) \geq P(X,Y)$$

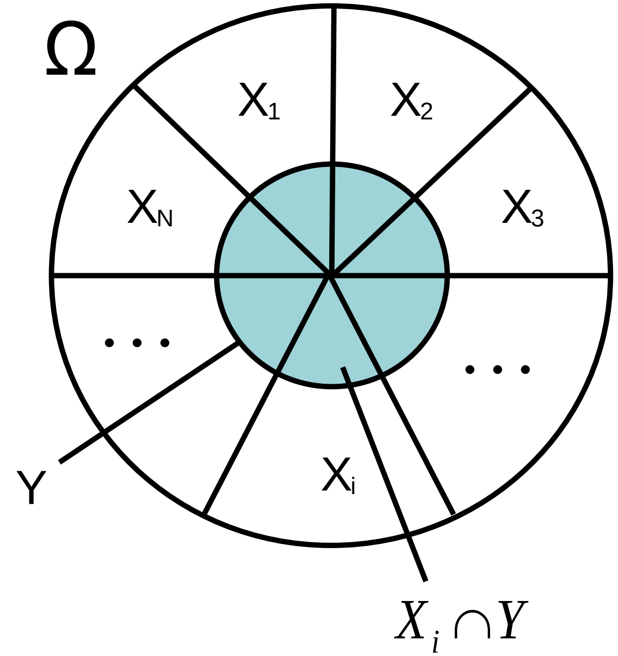
- **Statistische Unabhängigkeit**

$$P(X \mid Y) = P(X) \quad \text{v} \quad P(X,Y) = P(X)P(Y)$$

Statistics Fundamentals

❑ Vollständiges Ereignissystem

- $$P(Y) = \sum_{i=1}^{N} P(X_i, Y)$$



❑ Bayes Function

- $$P(X_i \mid Y) = \frac{P(Y \mid X_i) \cdot P(X_i)}{P(Y)} = \frac{P(Y \mid X_i) \cdot P(X_i)}{\sum_{k=1}^{N} P(Y \mid X_k) \cdot P(X_k)}$$

- **Distribution** (Verteilung)

X – discrete random variable

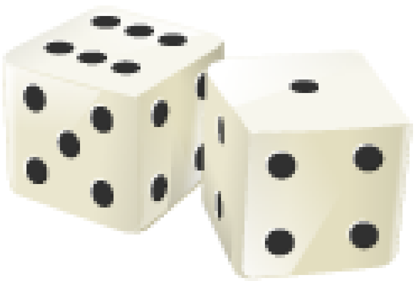
- Function $x(i) = P(X = i)$, $i = 0,1,2,\ldots,X_{max}$ (Distribution)
  - $x(i) \in [0,1]$
  - $\sum_{i=0}^{X_{max}} x(i) = 1$ (Vollständigkeitsrelation)

- Example:

  Rolling two dice
  - $\omega_1=\{2\}, \omega_2=\{3\}, \ldots, \omega_{11}=\{12\}$ => $\Omega = \{\omega_1, \omega_2, \ldots, \omega_{11}\}$
  - $X \in \{2, 3, 4, \ldots, 12\}$

Statistics Fundamentals

- Example: Throwing two dice



Sample Space (Ereignisraum)

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Statistics Fundamentals

- Example: Throwing two dice

Sample Space (Ereignisraum)

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
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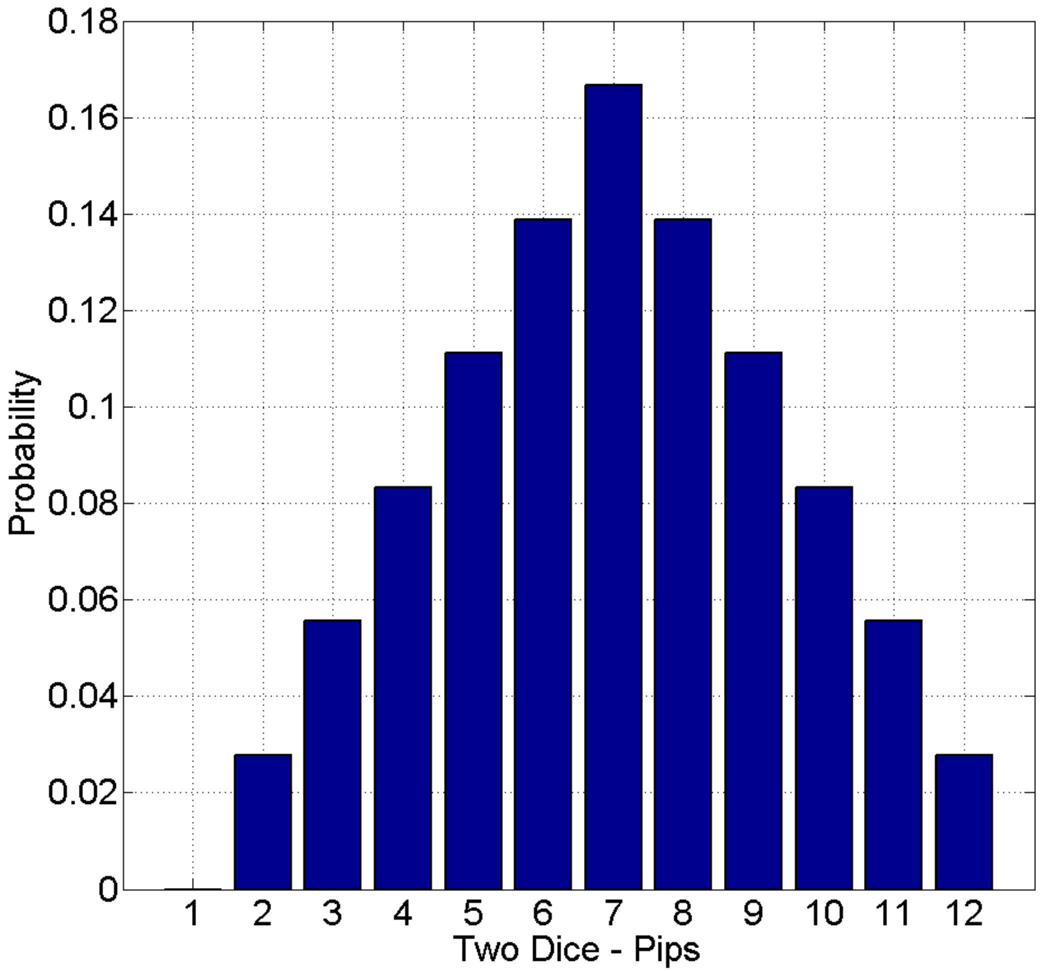


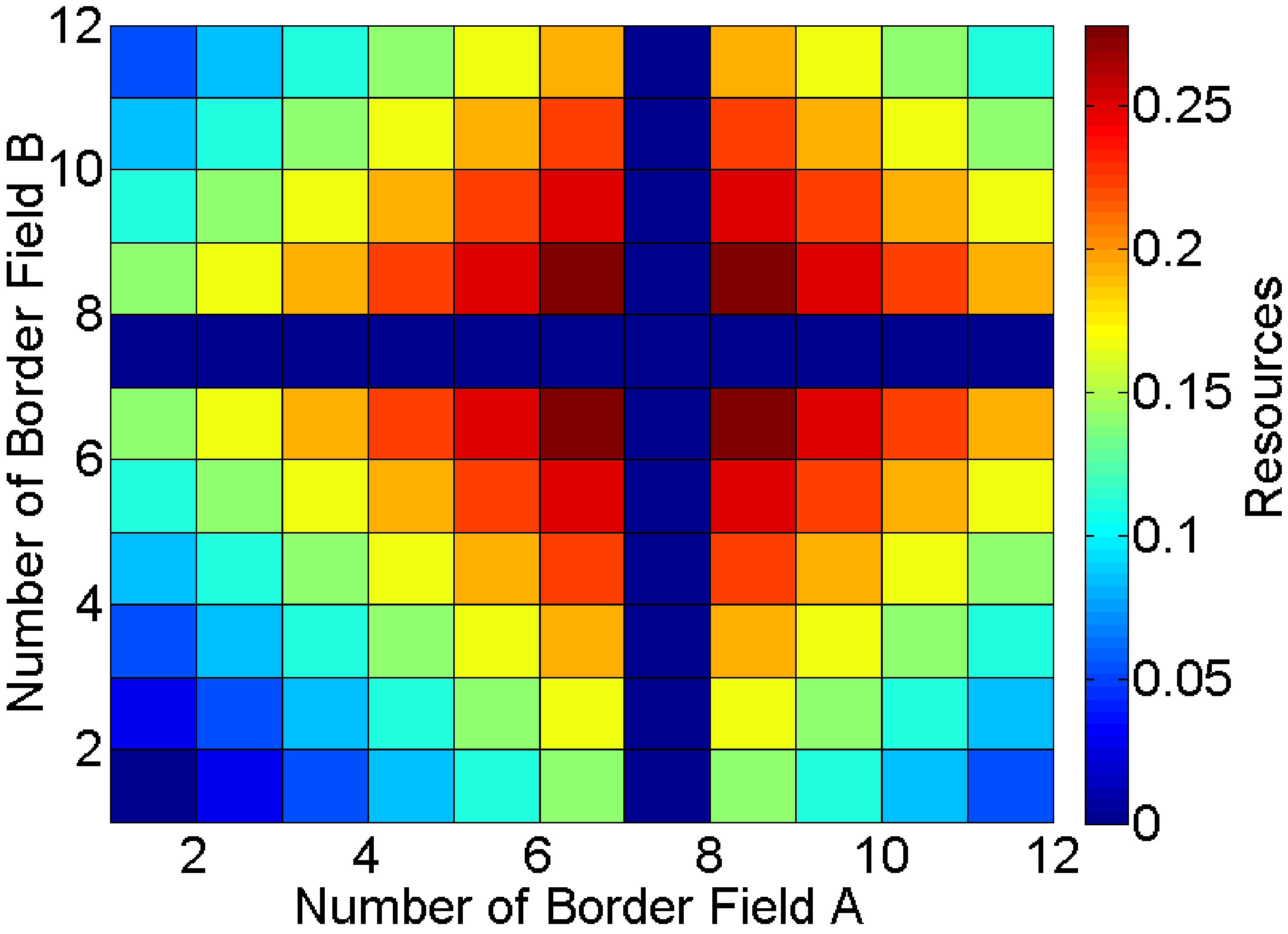
Distribution

Palour Game: Die Siedler von Catan

- Rules:
  - Players are only allowed to build along borders of a field
  - Players roll two dice
  - If the sum of the dice corresponds to the number of the field, the player gets the resources from this field

- Question
  - Where is the best place for a building?

Statistics Fundamentals



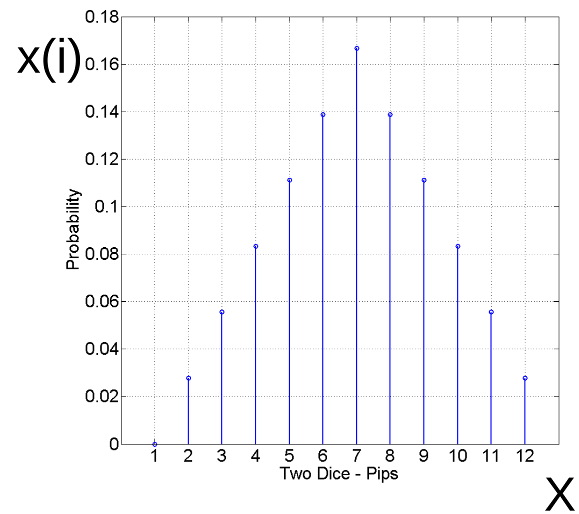
9
8
7
6
5
4
3
2
1

Prison

Are all fields reached with the same probability?

Statistics Fundamentals

- **Probability Mass Function** (Verteilung)
  - Discrete random variable X
  - i value of the random variable X
  - x(i) probability that the outcome of random variable X is i
  - $x(i) = P\{X = i\}, \quad i = 0,1,\ldots,X_{\max}$ (Distribution)
  - $\sum_{i=0}^{X_{\max}} x(i) = 1$ (Vollständigkeitsrelation)



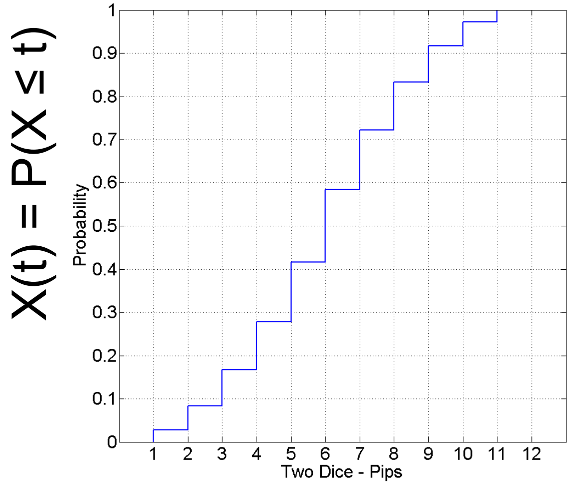
Probability Mass Function

Statistics Fundamentals

- **Cumulative Distribution Function** (Verteilungsfunktion)

$$X(t) = P\{X \le t\}$$

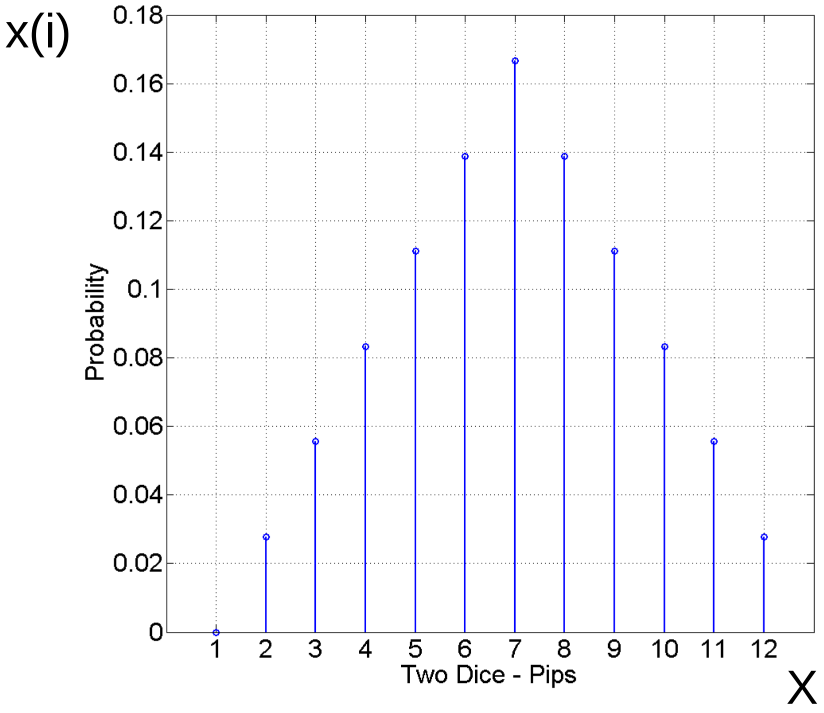
- $t_1 < t_2 \Longrightarrow X(t_1) \le X(t_2)$ (monotony)
- $t_1 < t_2 \Longrightarrow P\{t_1 < X \le t_2\} = X(t_2) - X(t_1)$
- $X(-\infty) = 0 \quad \wedge \quad X(\infty) = 1$
- $X^c(t) = 1 - X(t) = P\{X > t\}$



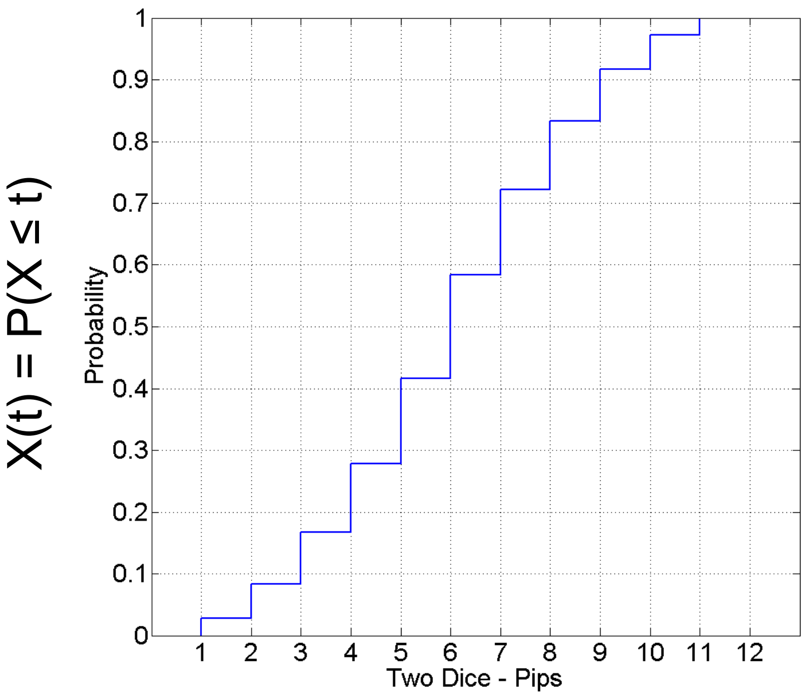
Cumulative Distribution Function

Statistics Fundamentals

Difference between probability mass function and cumulative distribution function



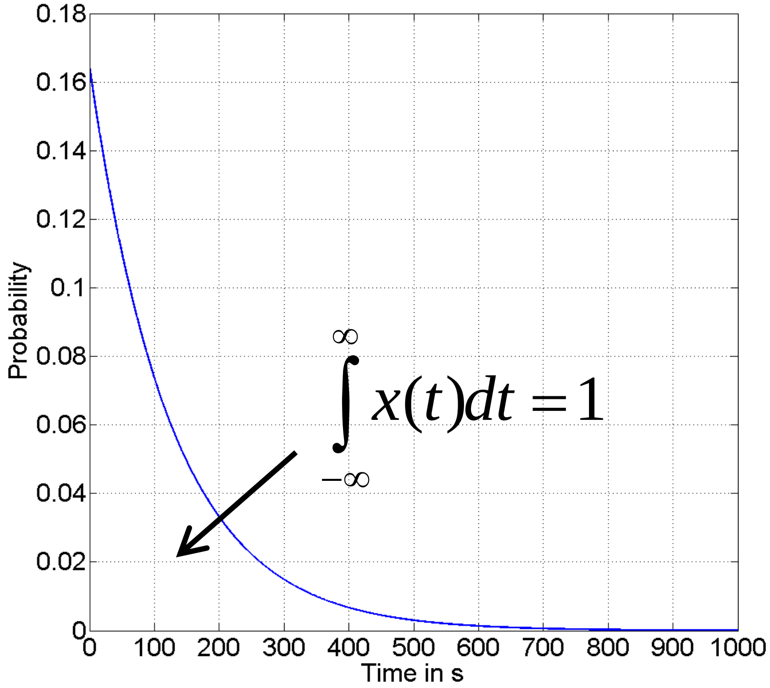
Probability Mass Function (Verteilung)



Cumulative Distribution Function (Verteilungsfunktion)

Statistics Fundamentals

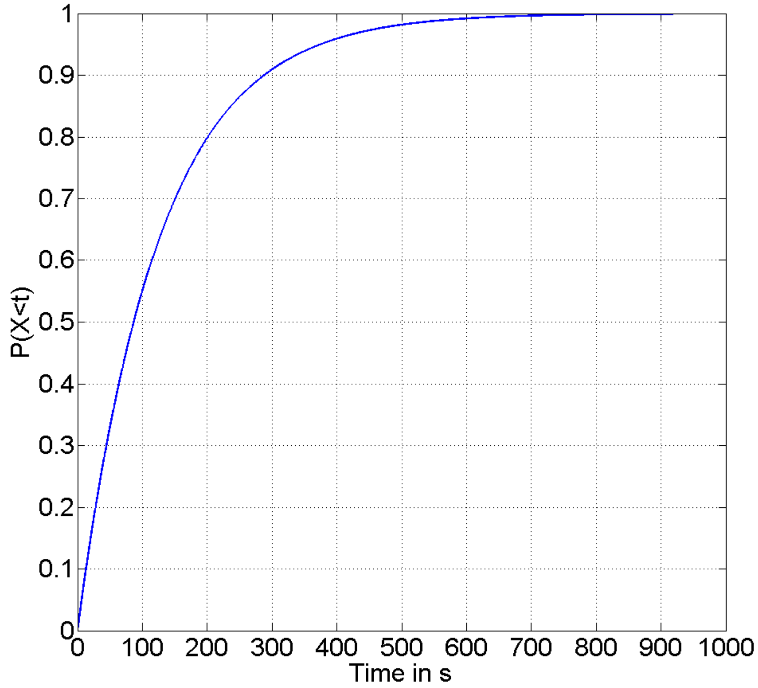
Continuous random variable



$$\int_{-\infty}^{\infty} x(t)dt = 1$$

Probability Density Function
(Verteilungsdichtefunktion)

$$x(t) = \frac{d}{dt} X(t)$$



Cumulative Density Function

$$X(t) = \int_{-\infty}^{t} x(t)dt$$

Statistics Fundamentals

- **Expectation** (Erwartungswert)
  - X : Probability density function
  - g(x) : Function of random variable X

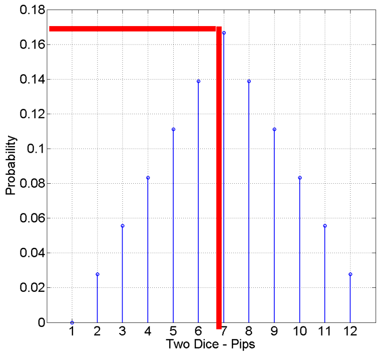
$$E[g(X)] = \int_{-\infty}^{\infty} g(t) \cdot x(t)\,dt$$

- **Mean** (Mittelwert einer Zufallsvariablen)

$$m_1 = E[X] = \int_{-\infty}^{\infty} t \cdot x(t)\,dt$$

- **Mode** (Outcome of the random variable with the highest probability)

$$c = Max(x(t))$$



Statistics Fundamentals

- Gewöhnliche Momente einer Zufallsvariablen
  - $$g(X) = X^k \Longrightarrow m_k = E\left[X^k\right] = \int_{-\infty}^{\infty} t^k \cdot x(t)dt, \qquad k = 0,1,2,\ldots$$

- Central moment (Zentrales Moment)
  - Variation of the random variable in respect to its mean

$$g(X) = (X - m_1)^k$$

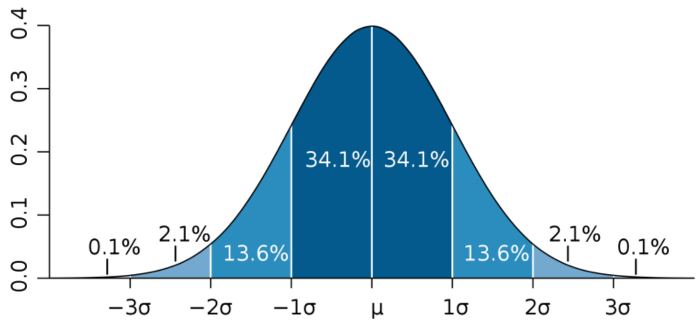
$$\Longrightarrow \mu_k = E\left[(X - m_1)^k\right] = \int_{-\infty}^{\infty} (t - m_1)^k \cdot x(t)dt, \qquad k = 0,1,2,\ldots$$

  - Special Case (k=2):

$$\mu_2 = E\left[(X - m_1)^2\right] = VAR[X]$$

Statistics Fundamentals

- **Standard deviation** (Standardabweichung)
  - $$\sigma_X = \sqrt{VAR[X]}$$



- **Coefficient of variation** (Variationskoeffizient)
  - $$c_X = \frac{\sigma_X}{E[X]}, \qquad E[X] > 0$$
  - The coefficient of variation is a normalized measure of dispersion of a probability distribution
  - It is a dimensionless number which does not require knowledge of the mean of the distribution in order to describe the distribution

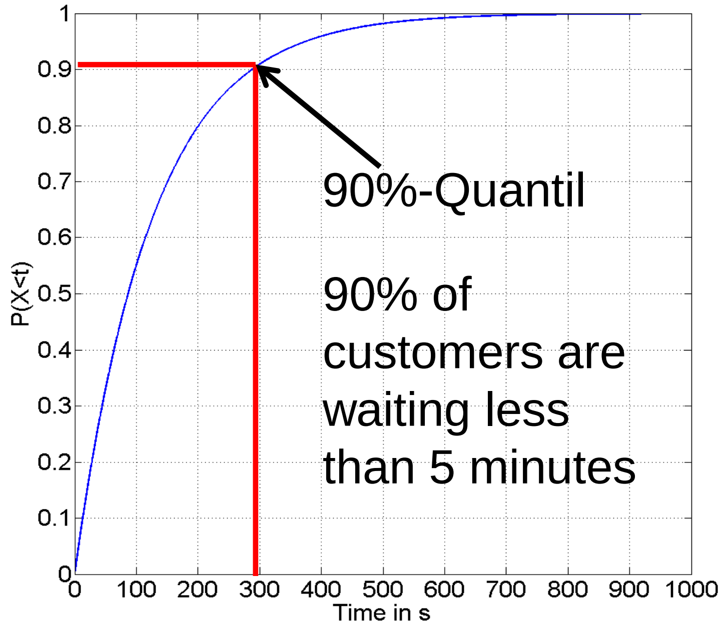
Picture taken from Wikipedia

Statistics Fundamentals

- **p-percentile t_p** (p-Quantil)

  A percentile is the value of a variable below which a certain percent of observations fall

  - VDF $F: R \rightarrow (0,1)$ (bijective)
  - $F(x) = P(X < x) = p$
  - $F^{-1}(x) = \inf\{x \in R : p \leq F(x)\}$
  - Special Case:
    - Median 0.5-percentile
    - Upper percentile 0.75-percentile
    - Lower percentile 0.25-percentile
  - Typical Use Case:
    - QoS in networks
      (e.g. 99.9%-percentile of the delay)



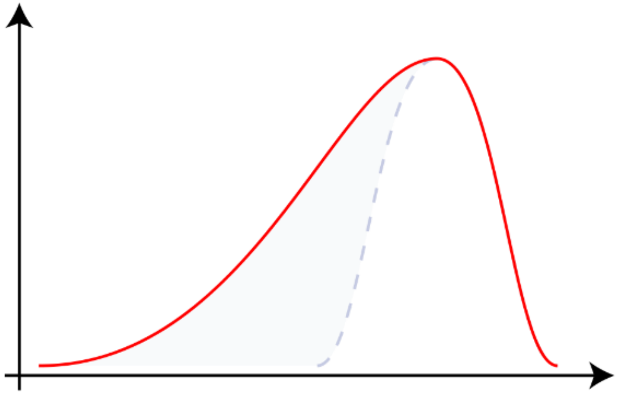
Cumulative Density Function

Statistics Fundamentals

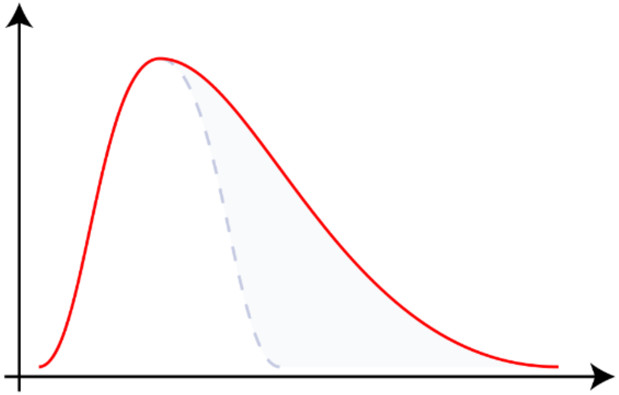
- **Skewness** (Schiefe)

  Skewness describes the asymmetry of a distribution
  - v < 0 : The left tail of the distribution is longer (linksschief)
    => Mass is concentrated in the right
  - v > 0 : The right tail of the distribution is longer (rechtsschief)
    => Mass is concentrated in the left

$$\upsilon_X = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{\mu_3}{\sigma^3}$$



Negative Skew



Positive Skew

Picture taken from Wikipedia

Statistics Fundamentals

Scalability Issues

- Multiplication of a random variable X with a scalar s
  - $Y = s \cdot X$
  - $E[Y] = s \cdot E[X]$
  - $VAR[Y] = s^2 \cdot VAR[X]$
- Addition of two random variables X and Y
  - $Z = X + Y$
  - $E[Z] = E[X] + E[Y]$
  - $VAR[Z] = VAR[X] + VAR[Y]$ (only if A and B independent)

Covariance

Covariance is a measure which describes how two variables change together

$$Cov(X,Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X] \cdot E[Y]$$

- Special Case: $Cov(X,X) = VAR[X]$
- Other Characteristics:
  - $Cov(X,a) = 0$
  - $Cov(X,Y) = Cov(Y,X)$
  - $Cov(aX,bY) = abCov(X,Y)$
  - $Cov(X+a,Y+b) = Cov(X,Y)$

Correlation function

Correlation function describes how two random variable tend to derivate from their expectation

$$Cor(X,Y)=\frac{Cov(X,Y)}{\sqrt{VAR(X)\cdot VAR(Y)}}$$

- Characteristics:
  - $Y=X \Rightarrow Cor(X,Y)=1$ (Maximum positive)
  - $Y=-X \Rightarrow Cor(X,Y)=-1$ (Maximum negative)
  - $Cor(X,Y)>0$ Both random variable tend to have either high or low values (difference to their expectation)
  - $Cor(X,Y)<0$ The random variables differ from each other such that one has high values while the other has low values and vice versa (difference to their expectation)

Statistics Fundamentals

- **Autocorrelation** (LK 4.9)
  - Autocorrelation is the cross-correlation of a signal with itself. In the context of statistics it represents a metric for the similarity between observations of a stochastic process. From a mathematical point of view, autocorrelation can be regarded as a tool for finding repeating patterns of a stochastic process.

  Definition:

  - Correlation of two samples with distance k from a stochastic process X is given by:

$$\Rightarrow \quad Cor(X,Y) \quad \text{with} \quad Y_i = X_{i+j}$$

  Use case:

  - Test of random number generators
  - Evaluation of simulation results (c.f. Batch-Means)

Example:

00101110101001101100010011101010100011

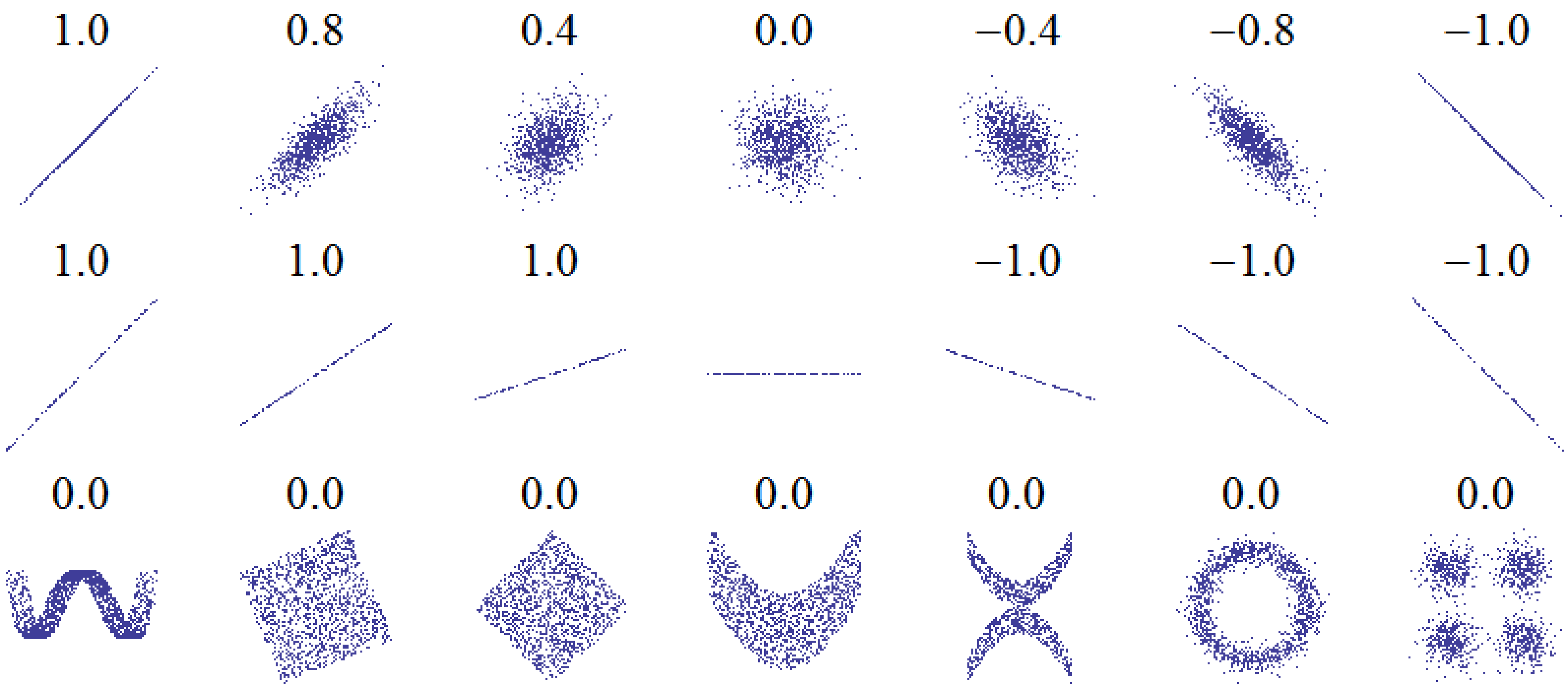
00101110101001101100010011101010100011

Random 

Autocorrelation Lag 4

Visualization of Correlation

Example: Two random variables X and Y are plotted against each other

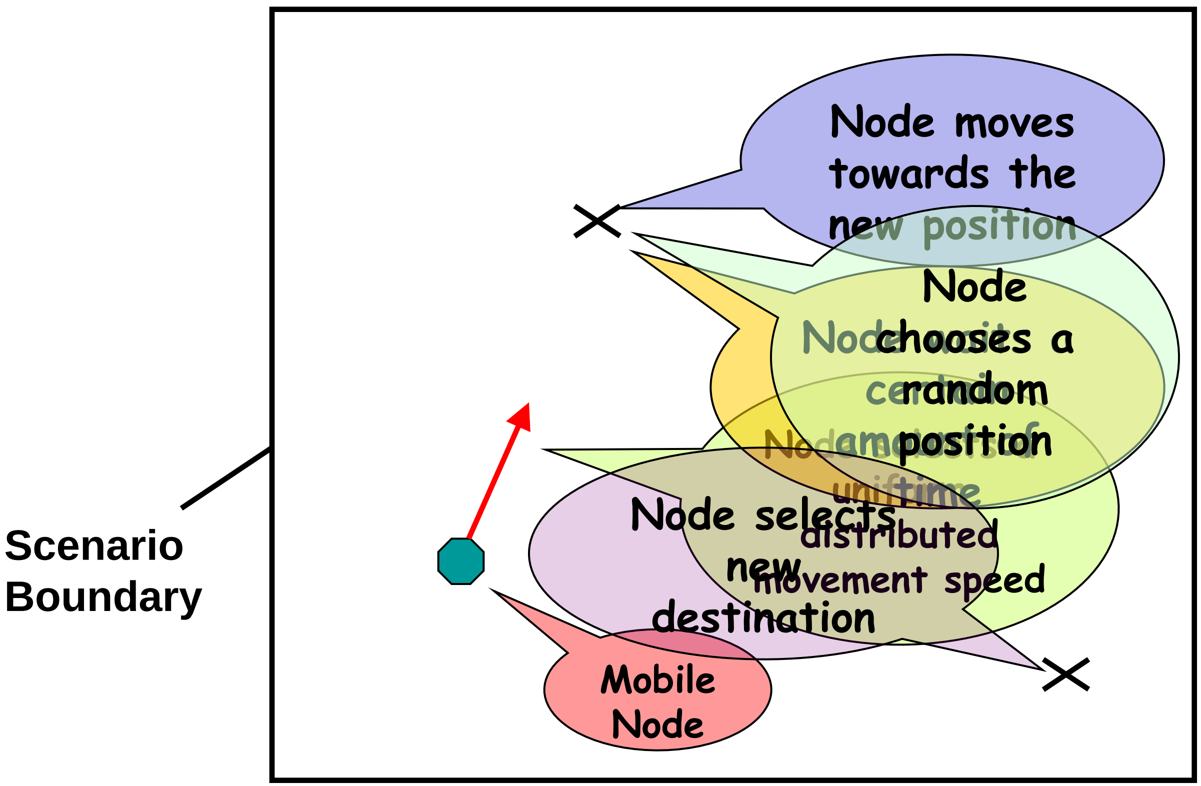


Picture taken from Wikipedia

Impact of correlation (1/2)

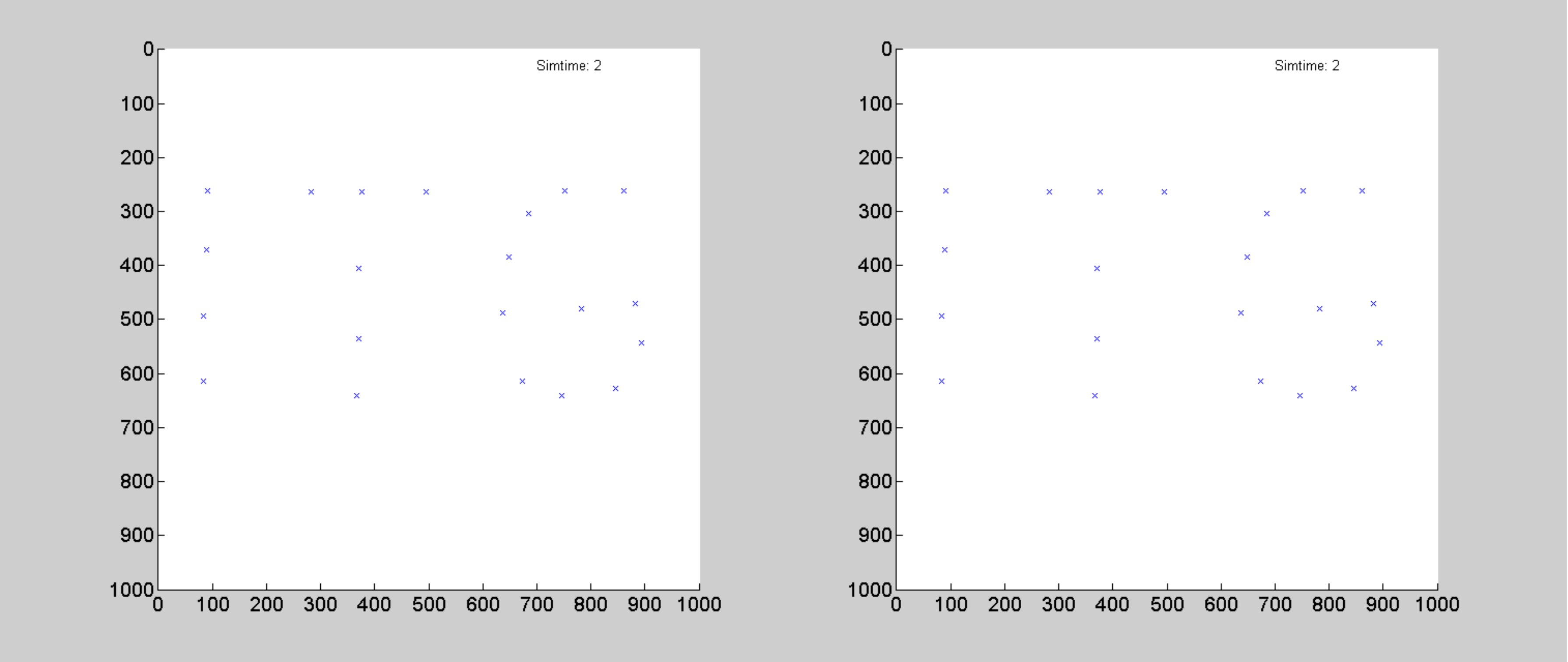
Example: Random Waypoint mobility model

Algorithm



Impact of correlation (2/2)

Example: Random Waypoint mobility model

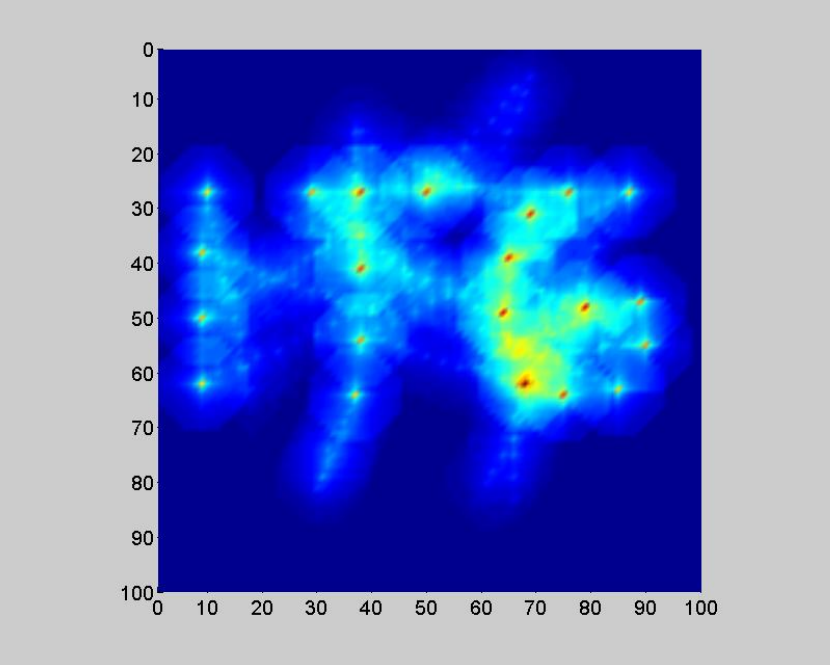


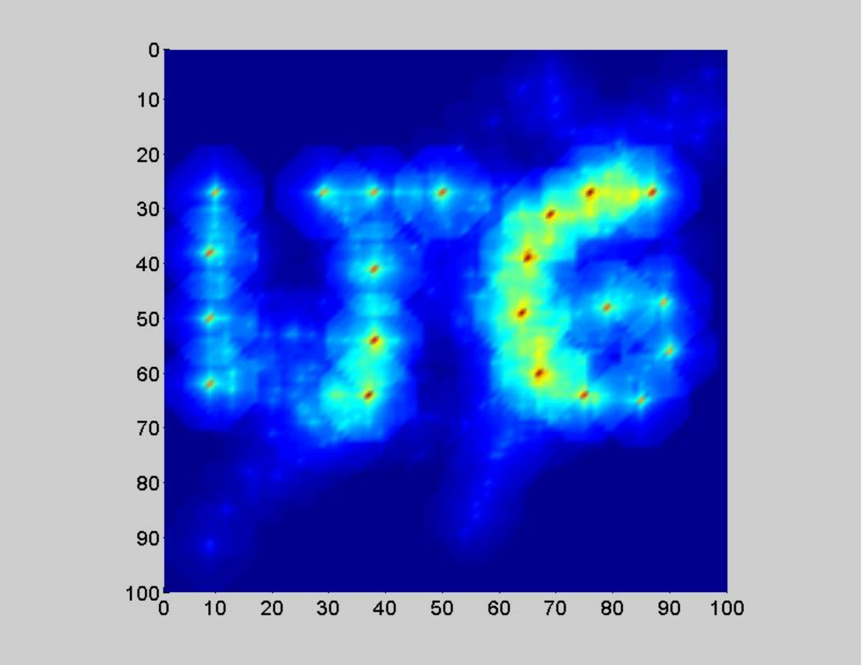
Uncorrelated next position selection

Correlated next position selection

Statistics Fundamentals

- Mobility Example





Statistics Fundamentals

- Visual comparison of different distributions

  - **Quantile-Quantile Plot**

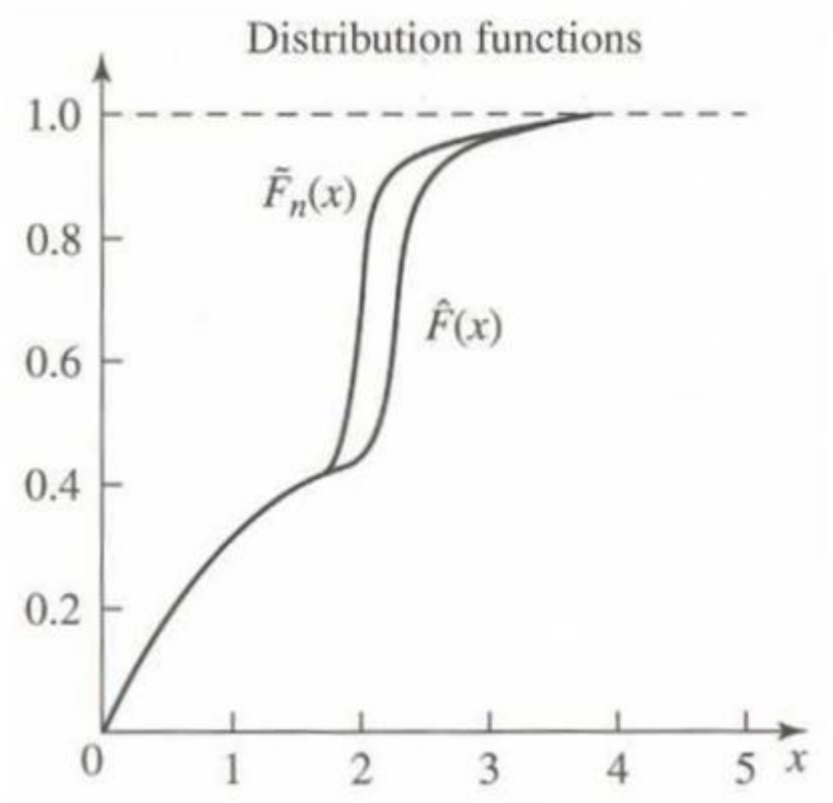
  - **Probability-Probability Plot**

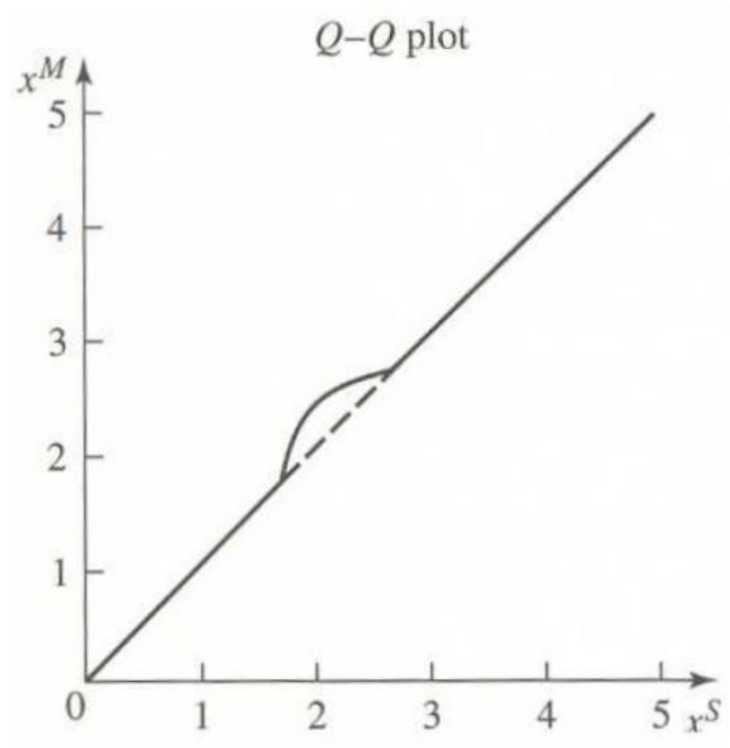
Statistics Fundamentals

Quantile-Quantile plots (QQ plots)

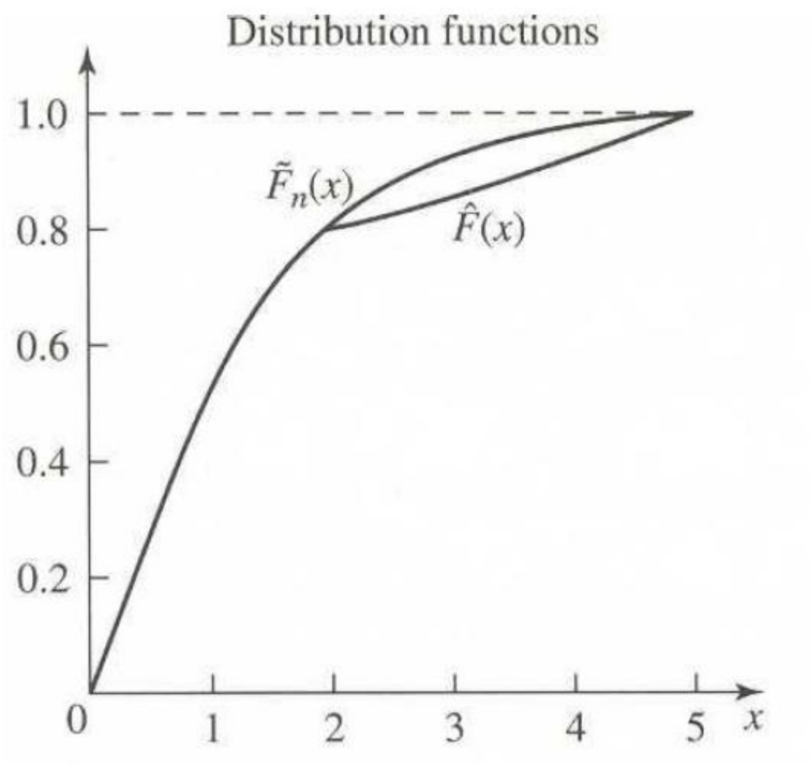
- Usage: Compare two distributions against each other
  - Usually: Measurement distribution vs. theoretical distribution – do the measurements fit an assumed underlying theoretical model?
  - Also possible: Measurement distribution vs. other measurement distribution – are the two measurement runs really from the same population, or is there variation between the two?
- How it works:
  - Determine 1% quantile, 2% quantile, …, 100% quantile for distributions
  - Plot 1% quantile vs. 1% quantile, 2% quantile vs 2% quantile, etc.
  - Not restricted to percentiles – usually, each of the *n* data points from the measurement is taken as its own 1/*n* quantile
- How to read:
  - If everything is located along the line *x*=*y* then the two distributions are very similar
  - QQ plots amplify discrepancies near the "tail" of the distributions
- Warning about scales:
  - Plot program often automatically assign X and Y different scales
  - Straight line indicates: choice of distribution OK, but parameters don't fit

- **QQ Plot**


Distribution functions
1.0
0.8
0.6
0.4
0.2
0
1
2
3
4
5
x
$\tilde{F}_n(x)$
$\hat{F}(x)$


Q–Q plot
x^M
5
4
3
2
1
0
1
2
3
4
5
x^S

- **QQ Plot**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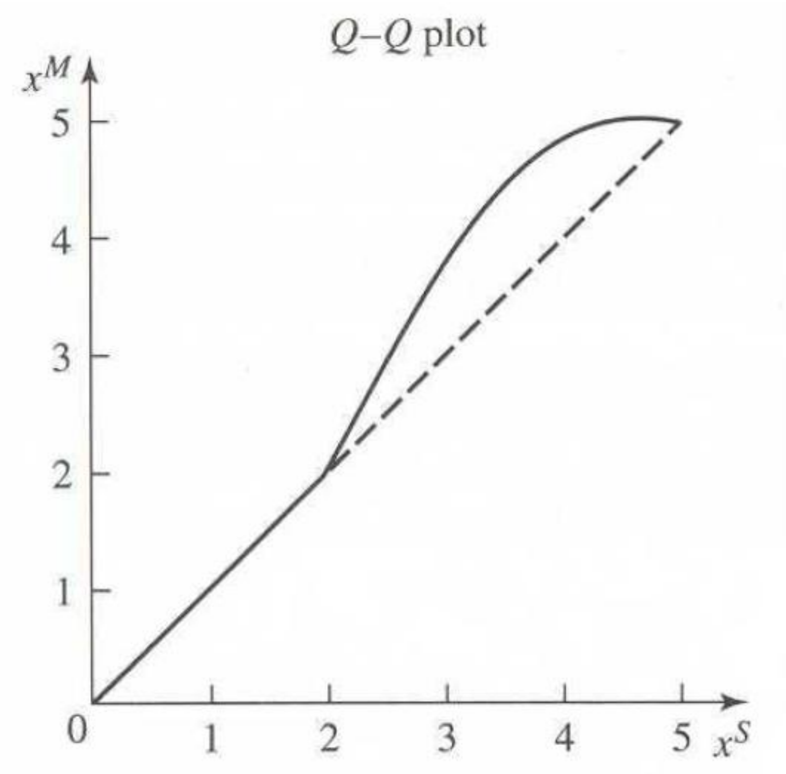

Q–Q plot
x^M
5
4
3
2
1
0
1
2
3
4
5
x^S

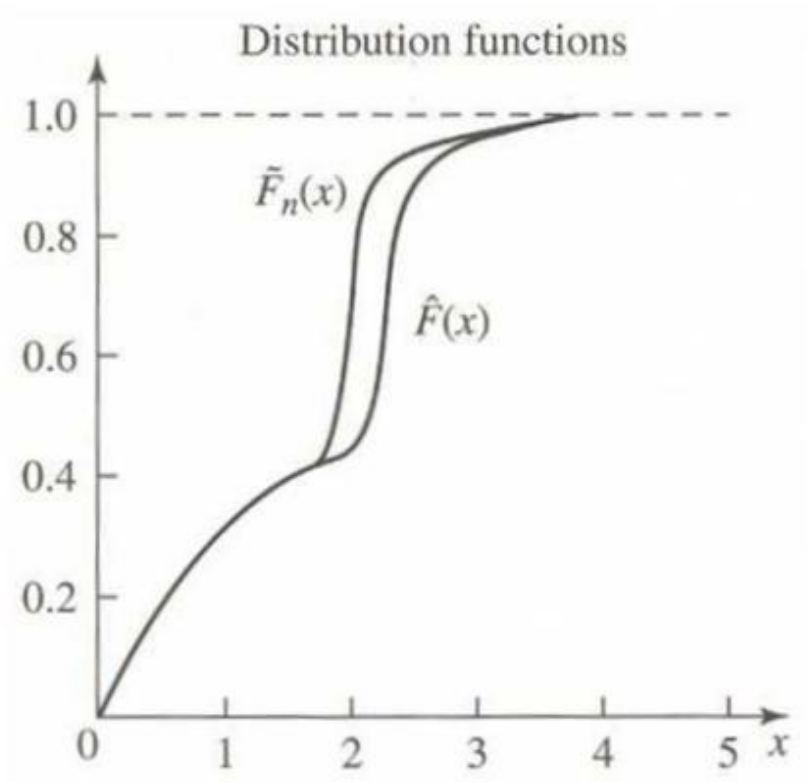
Probability-Probability plots (PP plots)

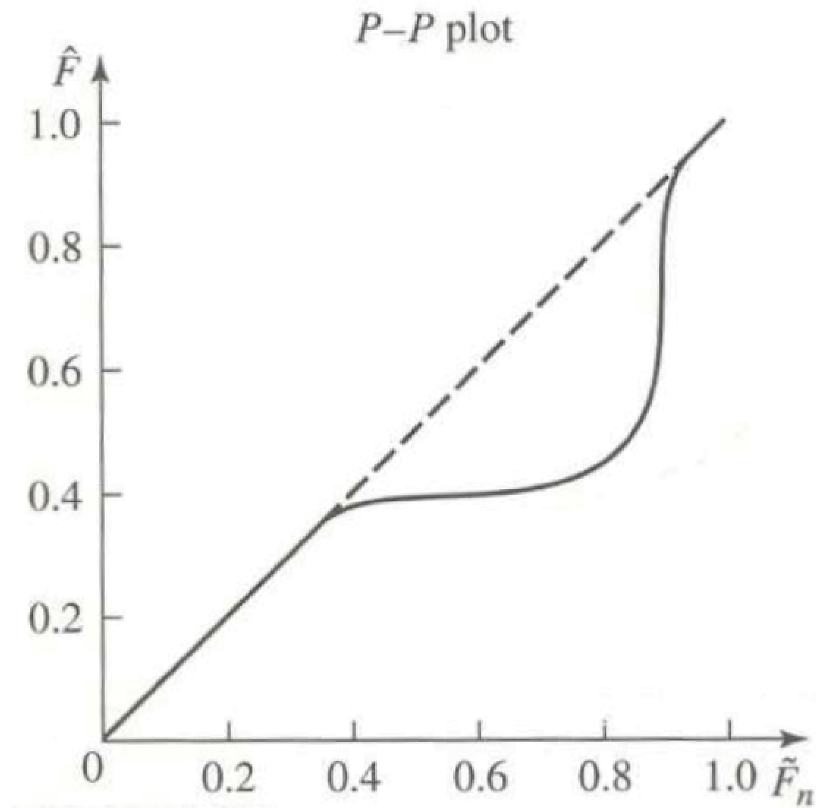
- Very similar to QQ plot
- QQ plot is more common, though

- Difference to QQ plot:
  - QQ plot compares [quantiles of] two distributions:
    1% quantile vs. 1% quantile, etc.
    - Graphically: the y axes of the cumulative density distribution functions are plotted against each other
  - PP plot compares probabilities of two distributions
    - Graphically: the y axes of the probability density functions are plotted against each other

- How to read:
  - Basically the same as QQ plot
  - PP plots highlight differences near the centers of the distributions
    (whereas QQ plots highlight differences near the ends of the distributions)

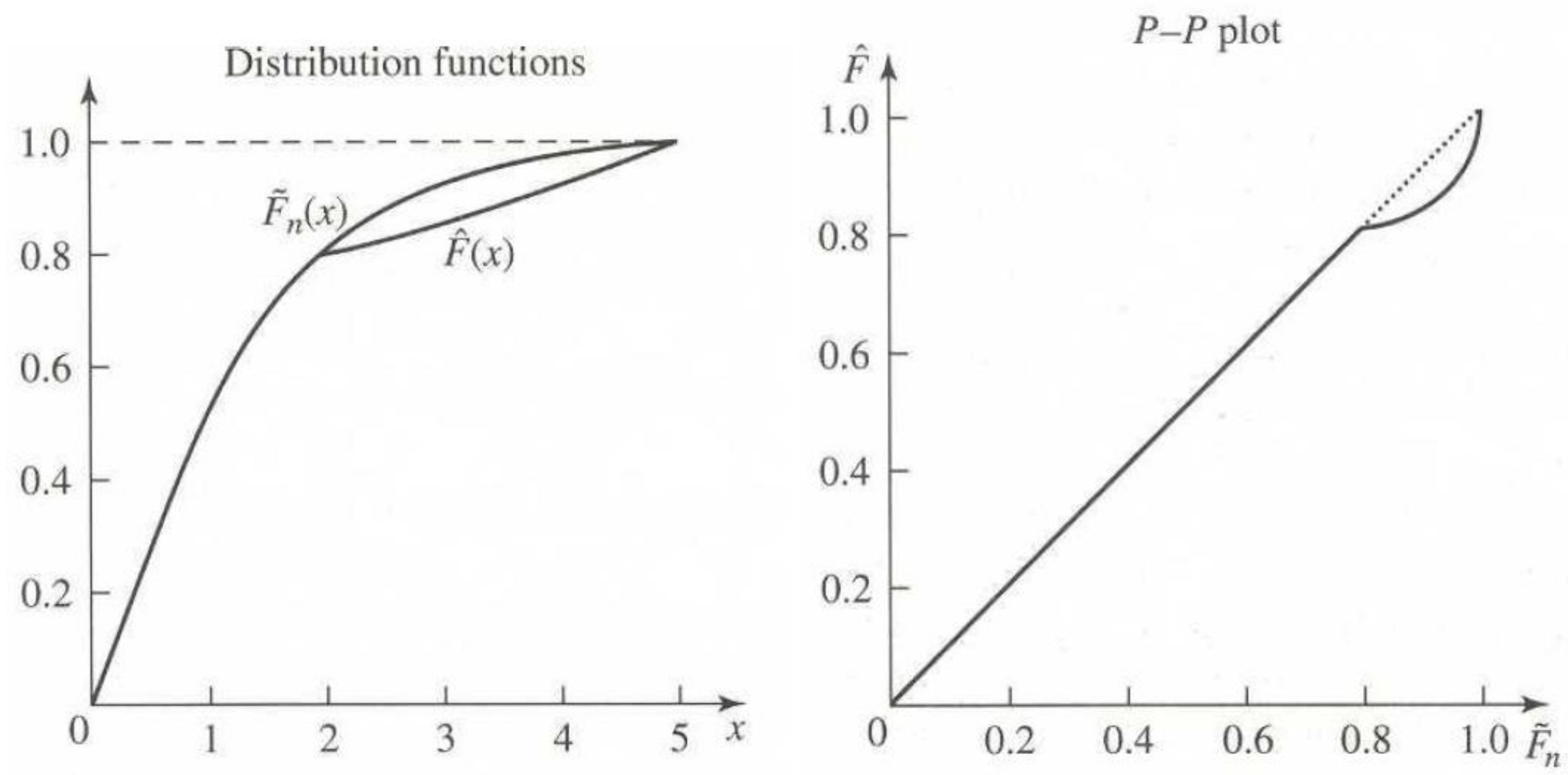
- **PP Plot**





The difference between the "middles" of $\hat{F}(x)$ and $\tilde{F}_n(x)$ amplified by the P–P plot.

PP Plot



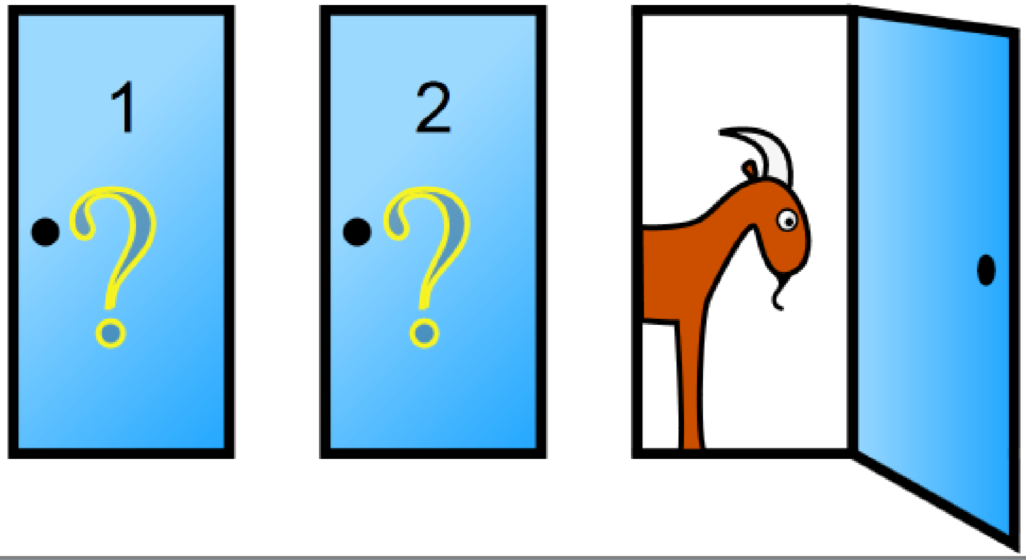
The difference between the right tails of $\hat{F}(x)$ and $\tilde{F}_n(x)$ amplified by the Q–Q plot.

- Monty Hall Problem – (also known as the goat problem)
  - American game show „Let‘s make a deal“ adopted in Germany „Geh auf Ganze“

Statistics Fundamentals

- Game rules:
  - Behind one door is a price
  - Behind the other doors is the goat / Zonk (It is assumed that the candidate is not interested in neither the goat nor the Zonk)
  - Candidate may choose one door
  - Game master will open one door after the decision of the candidate and will offer the candidate the choice to choose a different door.

Should the candidate change his/her decision?



Statistics Fundamentals

- Definition: RV Z=i: „Zonk/Goat is behind door i“
  - $P(Z=i)=1/n$ (Laplace)
- Definition: RV C=i: „Candidate has chosen door i“
  - $P(C=i)=1/n$ (Laplace)
- Z an C are independent
  - $P(Z=i \land C=i)=P(Z=i)\cdot P(C=i)$
- Definition: ZV O=i: „Door i was opened“
  - $P(O=i|Z=i)=0$ „The winning door was not opened“
  - $P(Z=i \land O=i) = P(O=i|Z=i)\cdot P(Z=i) = 0$ (Bayes)
  - $P(Z=i \land O\neq i) = P(Z=i) - P(Z=i \land O=i) = P(Z=i)$ (Totale Wahrscheinlichkeit)
- $P(O=i|C=i)=0$ „The selected door will NOT be opened“
  - $P(C=i \land O=i) = P(O=i|C=i)\cdot P(C=i) = 0$ (Bayes)
  - $P(C=i \land O\neq i) = P(C=i) - P(C=i \land O=i) = P(C=i)$ (Totale Wahrscheinlichkeit)

Statistics Fundamentals

- Win probability if the player does not change his selection

$$P(Z=i\,|\,C=i\wedge O\neq i)=\frac{P(Z=i\wedge(C=i\wedge O\neq i))}{P(C=i\wedge O\neq i)}$$

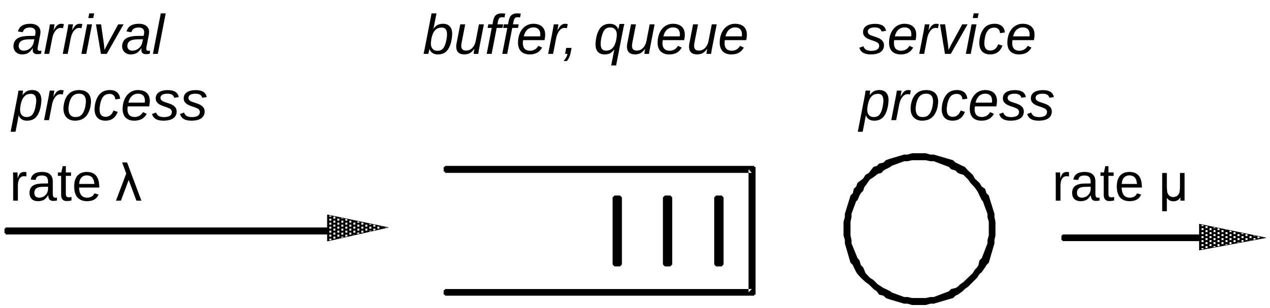
$$=\frac{P(Z=i\wedge C=i)}{P(Z=i)}=\frac{\frac{1}{n}\cdot\frac{1}{n}}{\frac{1}{n}}=\frac{1}{n}$$

- Win probability if player changes his/her selection

$$P(Z=i\,|\,C\neq i\wedge O\neq i)=\frac{1-P(Z=i\wedge(C=i\wedge O\neq i))}{n-2}$$

$$=\frac{1-\frac{1}{n}}{n-2}=\frac{n-1}{n\cdot(n-2)}$$

Waiting Queue Theory



3.4
0.6
1.7
1.6
0.7
0.7
1.3

inter-arrival times

random numbers

1.7
1.0
3.3
4.0
0.7
...

service times

Statistics Fundamentals

What are we talking about… and why?

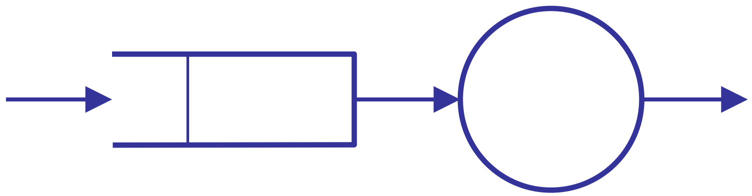
- Simple queue model:
  - Customers arrive at random times
  - Execution unit serves customers (random duration)
  - Only one customer at a time; others need to queue

- Standard example

- Give deeper understanding of important aspects, e.g.
  - Random distributions (input)
  - Measurements, time series (output)
  - …

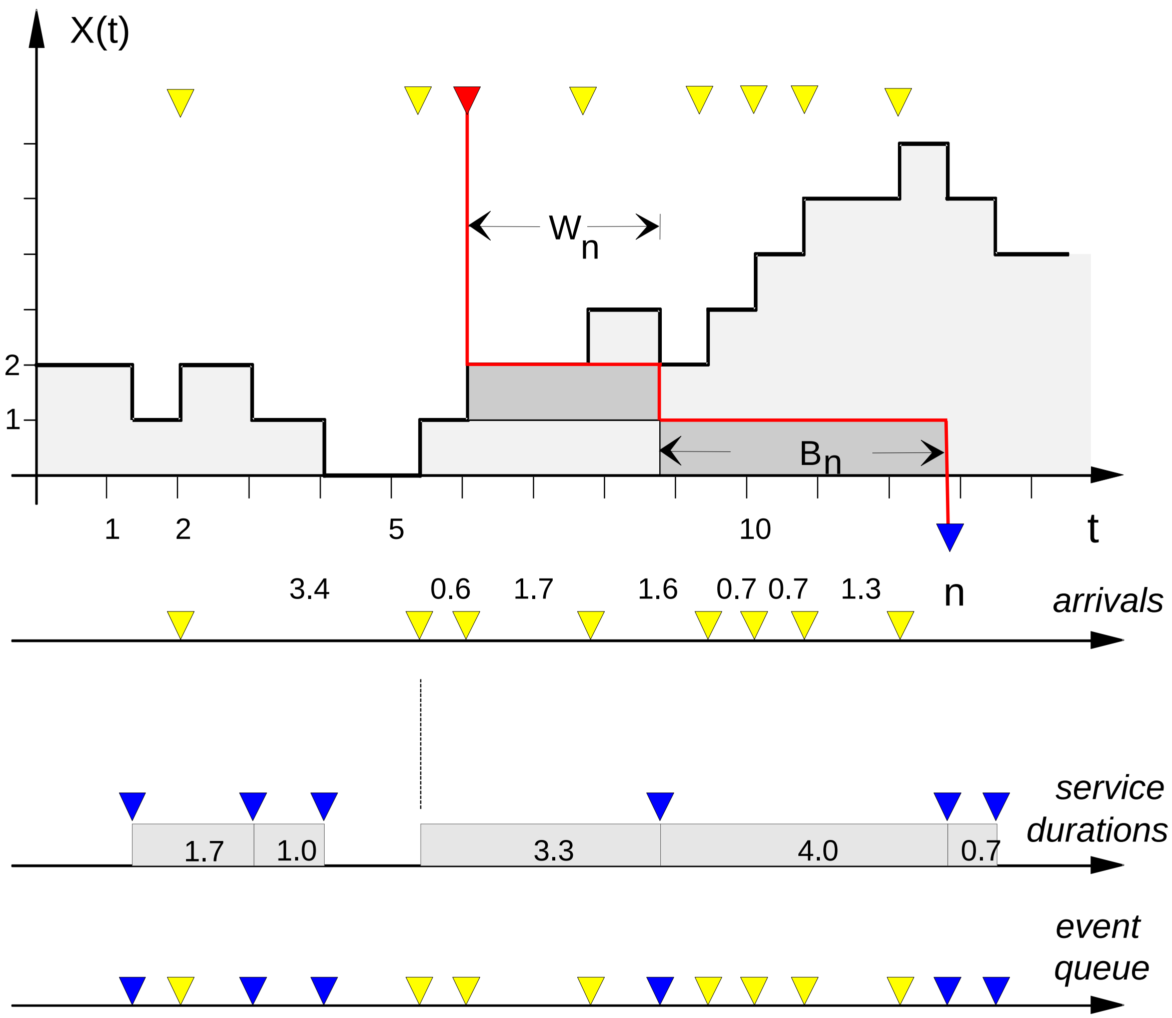
Queuing model: Input and output

- Input:
  - (Inter-)arrival times of customers (usually random)
  - Job durations (usually random)



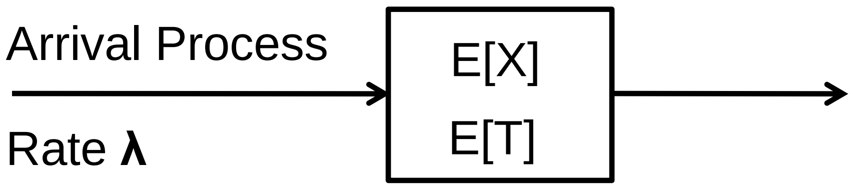
- Direct output:
  - Departure times of customers
- Indirect output:
  - Inter-arrival times for departure times of customers
  - Queue length
  - Waiting time in the queue
  - Load of service unit (how often idle, how often working)

Little Theorem



Little Theorem

- λ : average arrival rate
- E[X] : average number of packets in the system
- E[T] : average retention time of packets in the system



$$\boxed{t_o \to \infty}$$

$$\bar{T} = \frac{1}{N}\sum_{i=1}^{N} T_i \approx \frac{1}{N}\int_0^{t_o} X(t)dt$$

$$\lambda = \lim_{t_o \to \infty} \bar{\lambda} = \lim_{t_o \to \infty} \frac{N}{t_o}$$

$$\bar{X} = \frac{1}{t_o}\int_0^{t_o} X(t)dt \Longrightarrow \bar{X} \approx \frac{N}{t_o} E[T]$$

$$E[T] = \lim_{t_o \to \infty} \bar{T} = \lim_{t_o \to \infty} \frac{1}{N}\sum_{i=1}^{N} T_i$$

$$\bar{\lambda} \approx \frac{N}{t_o} \Longrightarrow \bar{\lambda} \cdot \bar{T} \approx \bar{X}$$

$$E[X] = \lim_{t_o \to \infty} \bar{X} = \lim_{t_o \to \infty} \frac{1}{t_o}\int_0^{t_o} X(t)dt$$

- **Kendall Notation**

$$GI^{[x]} / GI / n - S$$

- S — Number of Places in the Queue
  - $S = 0$ Loss/Blocking System
  - $S = \infty$ Waiting System
- n — Number of Servers
- GI (second) — Service Time Distribution
- $[x]$ — Batch Arrival Process
- GI (first) — Arrival Process

Statistics Fundamentals

Queuing Discipline

▪	FIFO / FCFS	First In First Out / First Come First Served
▪	LIFO / LCFS	Last In First Out / Last Come First Served
▪	SIRO	Service In Random
▪	PNPN	Priority-based Service
▪	EDF	Earliest Deadline First

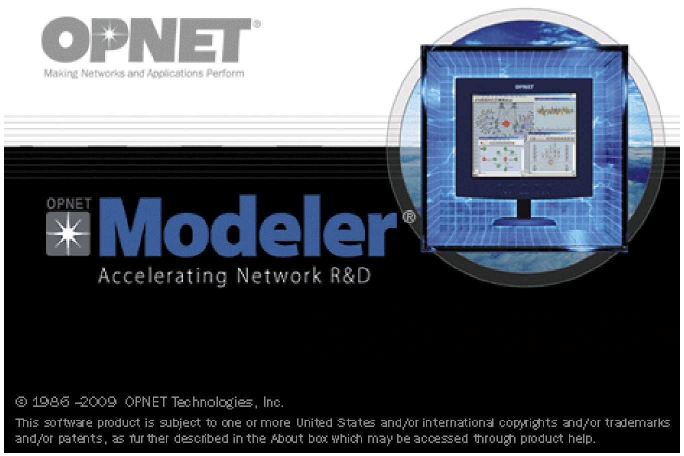
Distributions

▪	M	Markovian	Exponential Service Time
▪	D	Degenerate Distribution	A deterministic service time
▪	E_k	Erlang Distribution	Erlang k distribution
▪	GI	General distribution	General independent
▪	H_k	Hyper exponential	Hyper k distribution

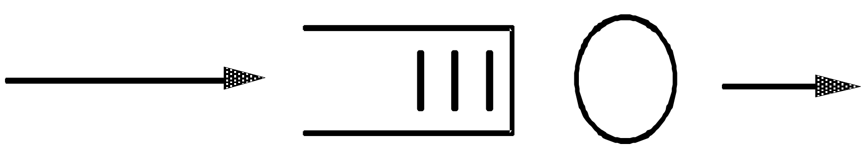
Statistics Fundamentals

- **System Characteristics**
  - Average customer waiting time
  - Average processing time of a customer
  - Average retention time of a customer
  - Average number of customers in the queue
  - Customer blocking probability
  - Utilization of the system / individual processing units

- Example

- 

How to model and evaluate waiting queues in OPNET



Statistics Fundamentals – Waiting Queue

- **Questions:**
  - How does the number of service units affect the system?
  - What impact has a higher variance of the arrival and/or service process on the performance of the system?
  - Which system has a higher utilization? One with an unlimited number of waiting slots or one with a limited number?
  - Which system has a lower retention time? One with many slow serving units or on with a single but fast serving unit?
  - How does the queuing strategy (FIFO, LIFO, EDF) affect the average waiting time and the waiting time distribution?

Statistics Fundamentals

- **Exercise**
  - System A: D / D / 1 - ∞
    - Arrival rate $\lambda = 1$ / s
    - Service rate $\mu = [1;10]$ / s
  - System B: M / M / 1 - ∞
    - Arrival rate $\lambda = 1$ / s
    - Service rate $\mu = [1;10]$ / s

- What is the maximum (meaningful) utilization of the system?
- Which system performs better?
- What impact does the utilization have on the system?

  - System C: M / M / 20 - ∞
    - Arrival rate $\lambda = 10$ / s
    - Service rate $\mu = 1$ / s
  - System D: M / M / 1 - ∞
    - Arrival rate $\lambda = 10$ / s
    - Service rate $\mu = 20$ / s

- Which system performs better?
- Would you prefer a single fast processing unit instead of multiple slow processing units?

Statistics Fundamentals

- **Exercise**
  - System E: M / M / 10 - ∞
    - Arrival rate $\lambda = 9$ / s
    - Service rate $\mu = 1$ / s
  - System F: M / M / 100 - ∞
    - Arrival rate $\lambda = 90$ / s
    - Service rate $\mu = 1$ / s

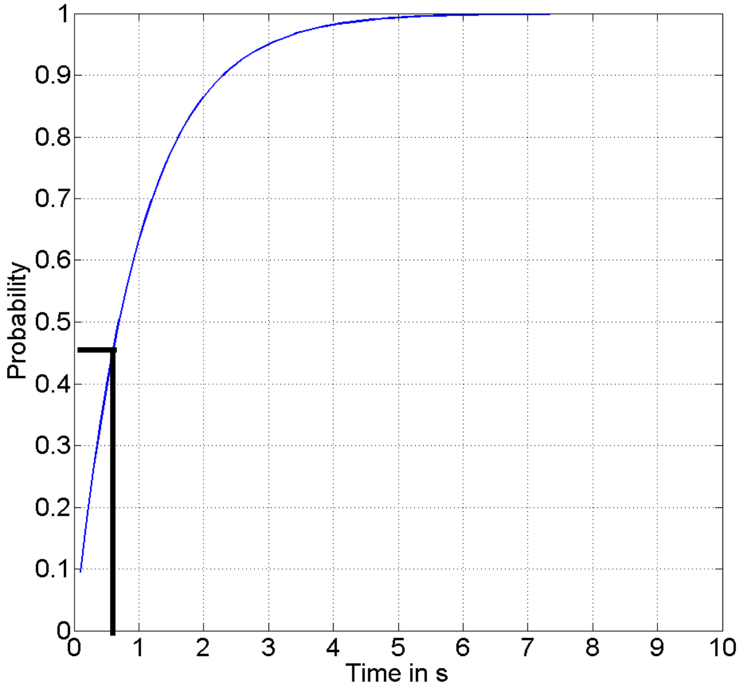
- What is the maximum (meaningful) utilization of the system?
- Which system performs better?

  - System G: M / D / 1 - ∞
    - Arrival rate $\lambda = 1$ / s
    - Service rate $\mu = 1 / 0.7$ / s
  - System H: D / M / 1 - ∞
    - Arrival rate $\lambda = 1$ / s
    - Service rate $\mu = 1 / 0.7$ / s

- Which system performs better?
- Which system has a shorter avg waiting time?

Statistics Fundamentals

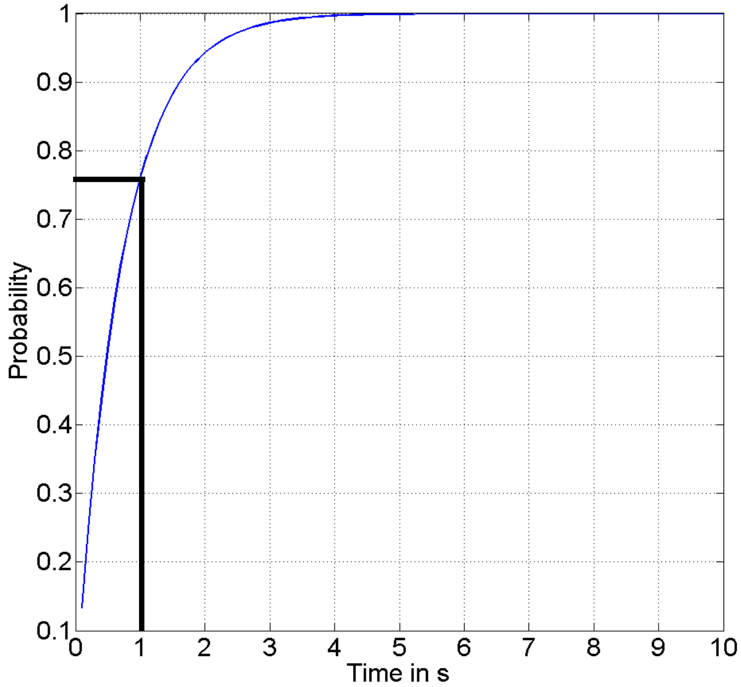
- System G: M / D / 1 - ∞



$\lambda = 1$

$$P(T > 0.7s) = 1 - P(T \leq 0.7s) = 0.55$$

- System H: D / M / 1 - ∞



$\mu = 1 / 0.7 = \sim 1.43$

$$P(T < 1s) = 0.75$$

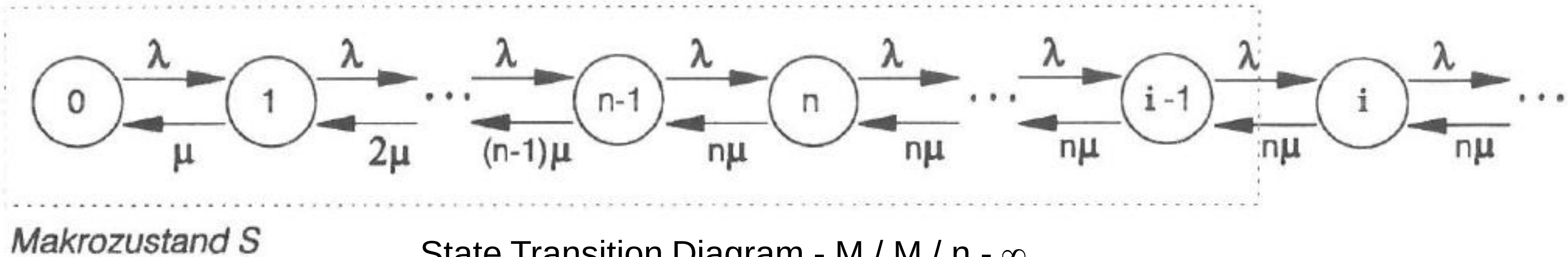
❑ M / M / n - ∞

- System of equations

$$\lambda x(i-1) = i\mu x(i), \qquad i = 1,2,3,\ldots,n,$$

$$\lambda x(i-1) = n\mu x(i), \qquad i = n+1,\ldots$$

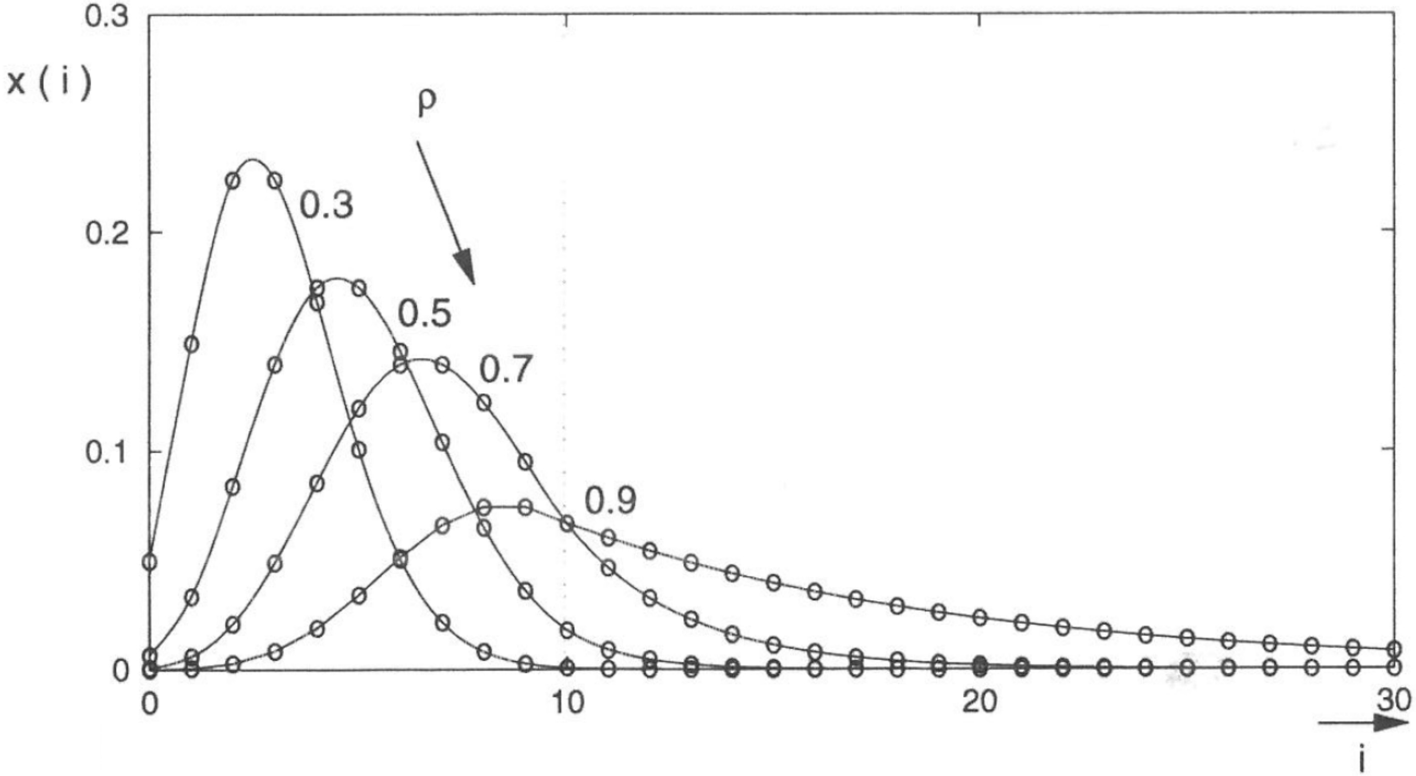
$$\sum_{i=0}^{\infty} x(i) = 1$$



State Transition Diagram - M / M / n - ∞

Picture taken from Tran-Gia, Analytische Leistungsbewertung verteilter Systeme, p. 98

- **M / M / 10 - ∞**

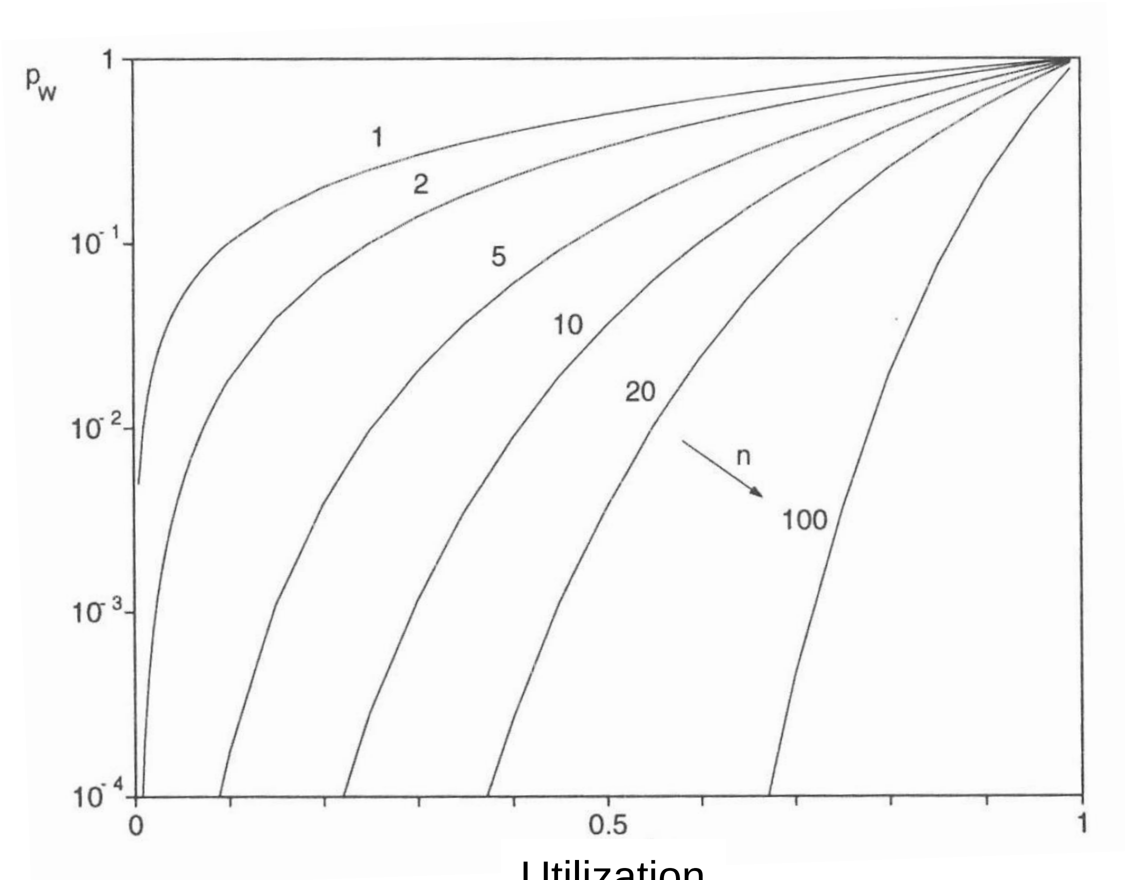


State Distribution - M / M / 10 - ∞

Picture taken from Tran-Gia, Analytische Leistungsbewertung verteilter Systeme, p. 99

Statistics Fundamentals

- **M / M / 10 - ∞**
  - **The waiting probability decreases with an increasing number of processing units (assuming constant utilization)**



Utilization

- n = number of processing units
- ρ_W - Waiting probability

Picture taken from Tran-Gia, Analytische Leistungsbewertung verteilter Systeme, p. 100