# Discrete Event Simulation

# IN2045

Dipl.-Inform. Alexander Klein
Dr. Nils Kammenhuber
Prof. Dr.-Ing Georg Carle

**Chair for Network Architectures and Services**
**Department of Computer Science**
**Technische Universität München**
**http://www.net.in.tum.de**

# Topics

- Generation of Random Variables
  - Inversion, Composition, Convolution, Accept-Reject
- Distributions – Continuous
  - Uniform, Normal, Triangle, Lognormal
  - Exponential, Erlang-k, Gamma,
- Distributions - Discrete
  - Uniform(discrete), Bernoulli, Geom, Poisson, General Discrete
- Random Number Generator (RNG)
- Linear Congruential Generator (LCG)
- $X^2$ Test
- Serial Test
- Spectral Test
- Shift Register
- Generalized Feedback Shift Register
- Mersenne Twister

Chapter is based on LK 6+8)

# Introduction - Random variates

- **Generation of U(0,1) random numbers**
  - Generation approaches

  - “Real”, “natural” random numbers: sampling from radioactive material or white noise from electronic circuits, throwing dice, drawing from an urn, …
    - Problems:
      - If used online: not reproducible
      - Tables: uncomfortable, not enough samples
  - USB – Random Number Generator – Developed at TUM

    http://www.heise.de/newsticker/meldung/Appliance-liefert-50-Millionen-Zufallsbits-pro-Sekunde-1125288.html

  - Pseudo random numbers: recursive arithmetic formulas with a given starting value (seed)
    - in hardware: shift register with feedback (based on primitive polynomials as feedback patterns)
    - in software: linear congruential generator (LCG) (Lehmer, 1951), …

# Generating random variates

- All algorithms are based on U(0,1) random variates

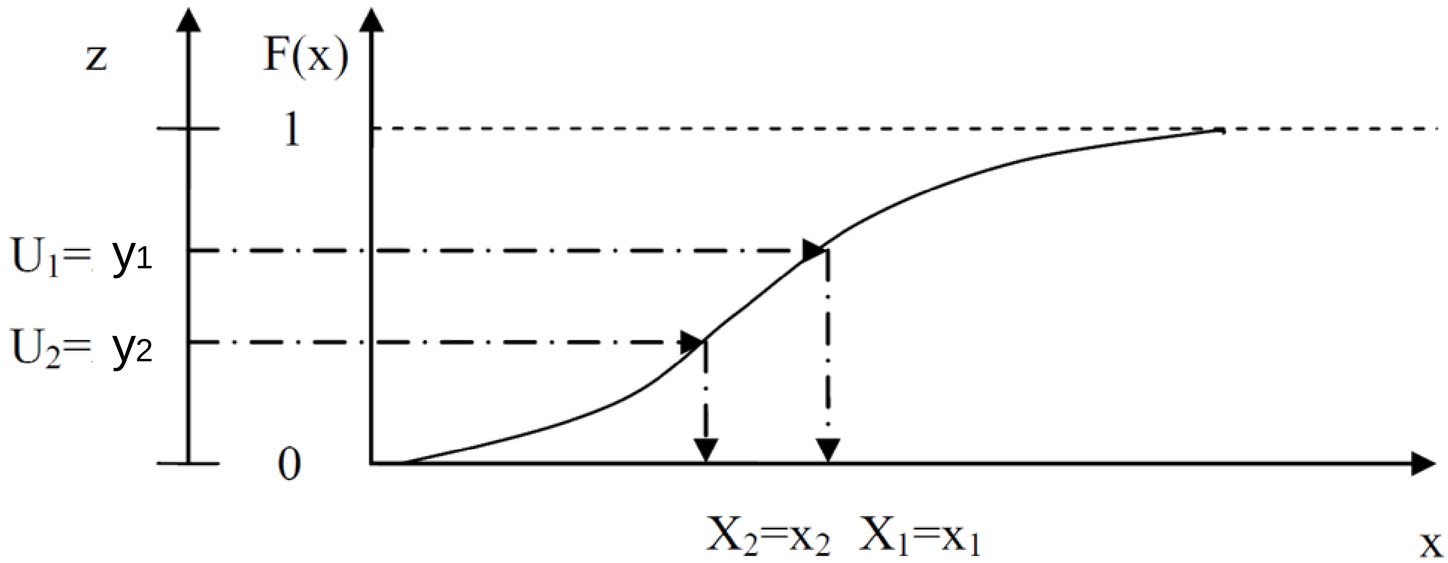
- Selection criteria
  - Exactness (generation of the desired distribution)
  - Efficiency
    - Storage requirements (large tables required?)
    - Execution time
      - Marginal execution time (for each sample)
      - Setup time (at start time)
  - Robustness (characteristics do not change for different parameters)
  - Complexity (you have to understand before you implement it)

- Huge literature available

# Random variates

- **Measurement**
  - Samples of a random variable X
  - What is the distribution function of random variable X?
- **Simulation**
  - Distribution function of the random variable is known in advance
  - How to generate samples which follow the distribution of the random variable?
- **Idea**
  - Generation of uniform distributed random numbers U(0,1) (Random number generator)
  - Transformation of the generated numbers according to the desired distribution of the random variable

# Inversion (LK 8.2)




- Random variable $y_i \sim U(0,1)$
- Transformation of $y_i$ according to a distribution function F(x) in a random variable $X_i$
  - $y_i = F(x_i) \rightarrow x_i = F^{-1}(y_i)$

# Inversion (LK 8.2)

Example: Generation of an exponential distribution with a mean value of λ

- Algorithm:
  - Generate U~U(0,1) (pseudo random numbers)
  - Return $X = F^{-1}(U)$

- Random variable $y_i$ ~ U(0,1)
- Transformation of $y_i$ according to a distribution function F(x) in a random variable $X_i$

$$F(x) = \begin{cases} 1 - e^{-\frac{x}{\lambda}} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F^{-1}(u) = -\lambda \ln(1-u) \quad \xrightarrow{\text{symmetry}} \quad F^{-1}(u) = -\lambda \ln u$$

# Composition

- Desired distribution function expressed as a convex combination of other distribution function

$$F(x) = \sum_{j=1}^{\infty} p_j F_j(x) \qquad where \qquad p \geq 0, \sum_{j=1}^{\infty} p_j = 1$$

  - Generate positive random integer J

$$P(J = j) = p_j \qquad for \;\; j = 1,2,....$$

  - Return X with distribution function $F_J$

# Convolution

- Desired random variable can be described as the sum of other random variable
  - 1. Generate $Y_1, Y_2, Y_3, \ldots, Y_k$
  - Return $X = Y_1 + Y_2 + Y_3 + \cdots + Y_k$

- Example:
  - k- Erlang distributed random variable with a mean ε can be expressed as the sum of k exponential random variables with a common mean k/ε

- Advantage: simple and intuitive approach
- Disadvantage: slow since multiple random number have to be generated in order to get a single sample

# Accept-Reject-Method (LK 8.2.4)

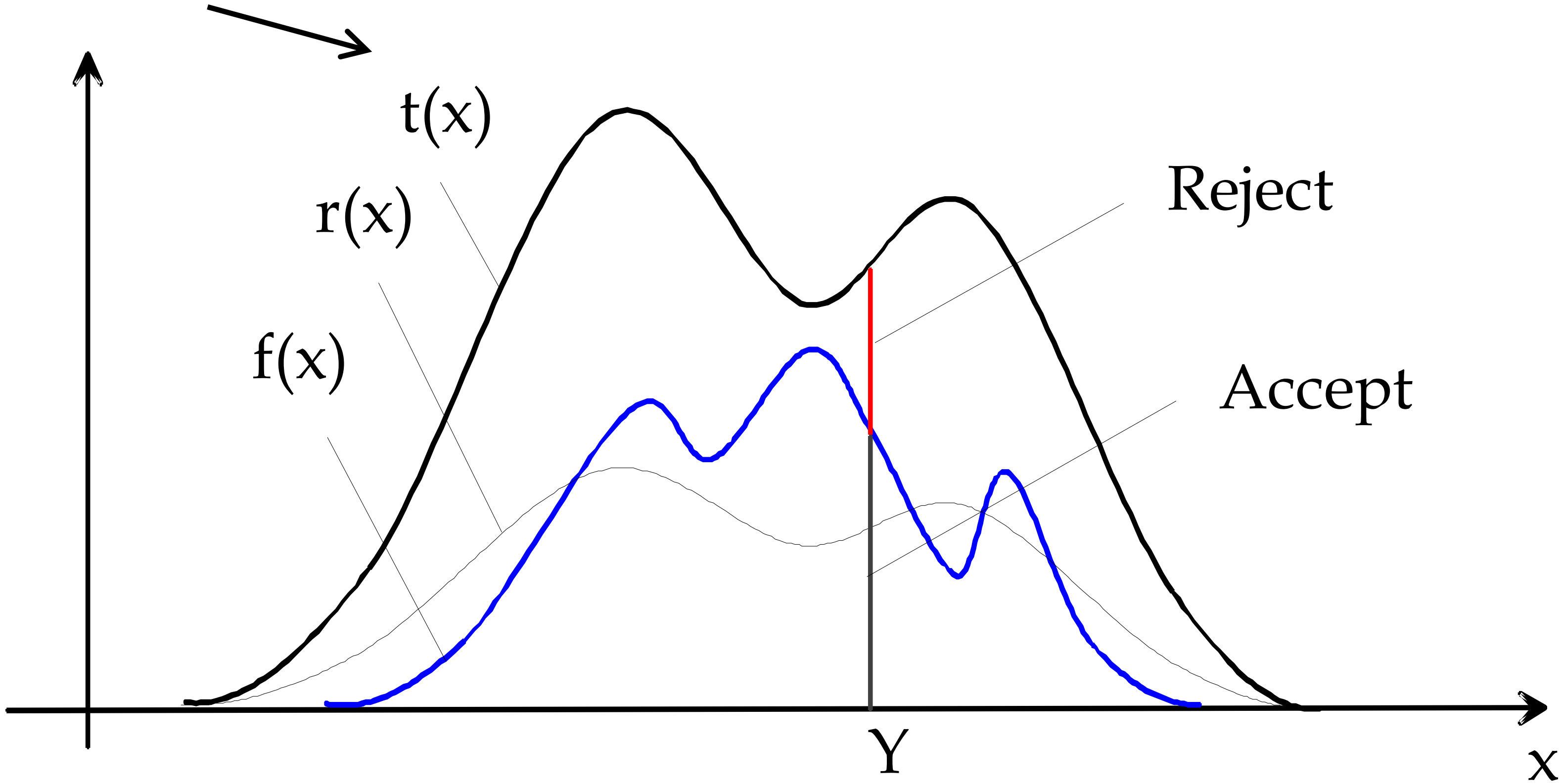
- Inverse transform, combination, and convolution are **direct** methods (work directly with the distribution function)

- Accept-Reject is used when other methods fail or are inefficient

- Density function is complex $\rightarrow$ select a “simpler” density function *r*

# Accept-Reject-Method (LK 8.2.4)

- Geometrical interpretation
  *Y* will be accepted if the point $(Y, U \cdot t(Y))$ falls under the curve *f* .
- The acceptance probability is high if *t(Y)-f(Y)* is small.
- Majorante von f(x) ⇨ $\forall x : t(x) \geq f(x)$



# Accept-Reject-Method (LK 8.2.4)

- **Indirect approach:**
- Preparation:
  - We need a function *t* that **majorizes** density *f*

$$t(x) \geq f(x) \quad \text{for all } x$$

$$c = \int_{-\infty}^{\infty} t(x)dx \geq \int_{-\infty}^{\infty} f(x)dx = 1$$

  - We obtain a density *r* by $r(x) = \frac{t(x)}{c}$

- **Algorithm**

1. Generate a random variable Y according to a density r

2. Generate a random number $U \sim U(0,1)$ (independent of Y)

3. Return $X = Y$ if $U \leq \frac{f(Y)}{t(Y)}$ **(ACCEPT)**

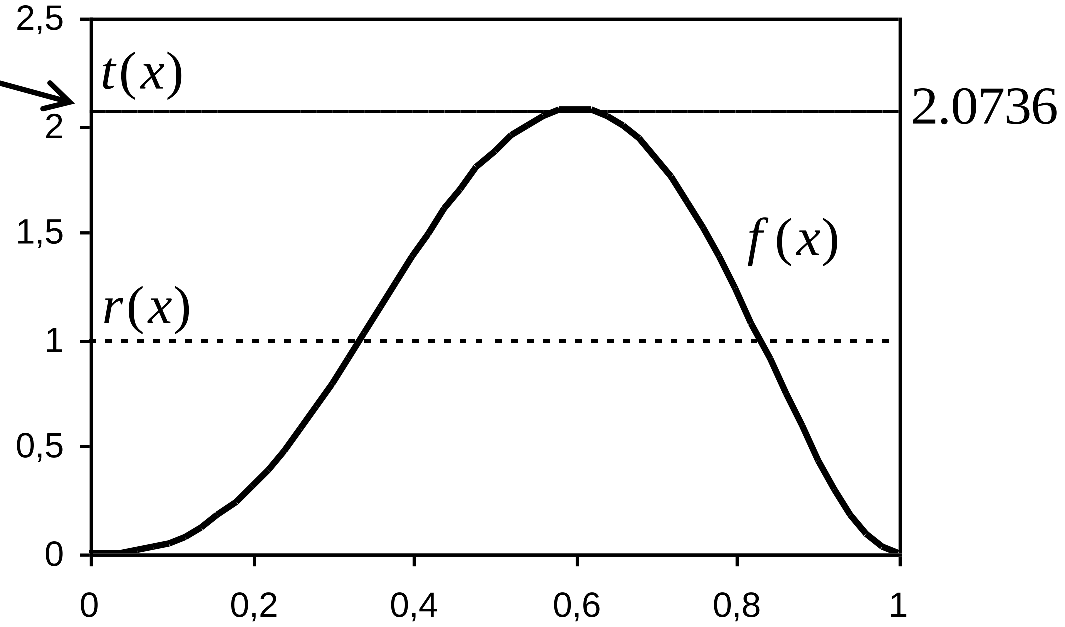
   Otherwise, go back to step 1 and try again **(REJECT)**

# Accept-Reject-Method (LK 8.2.4)

- Example: beta(4,3) distribution (6th order polynomial, hard to invert)

$$f(x) = \begin{cases} 60x^3(1-x)^2 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Majorante
von f(x)



# Accept-Reject-Method (LK 8.2.4)

- Efficiency:
  - Depends on the majorant series (x)
  - Probability of acceptance is 1/c ⇒ Average number of iterations
- Advantage:
  - Works for arbitrary density functions
- Disadvantage:
  - Number of required U(0,1) random numbers depends on the generated numbers (may causes problems with some statistics and may result variations of the simulation duration)
  - Requires at two U(0,1) random numbers in each iterations

# Random number generation

How to generate random numbers according to different distributions?

# Random numbers - Continuous

- **Uniform distribution**: $RV\ \ X \sim U(a,b)$ (LK 8.3.1)
  - Density function: $f(x) = \frac{1}{b-a}, X \in [a;b]$
  - Range: $[a;b]$
  - Distribution function: $F(x) = \frac{x-a}{b-a}$
  - Expectation: $E(X) = \frac{a+b}{2}$
  - Variance: $VAR(X) = \frac{(b-a)^2}{12}$
  - Generation: $U \sim U(0,1), X = a + (b-a)U$

# Random numbers - Continuous

- **Triangle distribution** (1/3): $RV \quad X \sim triang(a,b,c)$ (LK 8.3.15)

  - Density function:

$$f(x) = \begin{cases} \dfrac{2\cdot(x-a)}{(b-a)\cdot(c-a)} & if \quad a \le x \le c \\ \dfrac{2\cdot(b-x)}{(b-a)\cdot(b-c)} & if \quad c \le x \le b \\ 0 & otherwise \end{cases}$$

  - Distribution function:

$$f(x) = \begin{cases} 0 & if \quad x < a \\ \dfrac{(x-a)^2}{(b-a)\cdot(c-a)} & if \quad a \le x \le c \\ 1-\dfrac{(b-x)^2}{(b-a)\cdot(b-c)} & if \quad c \le x \le b \\ 1 & if \quad b < x \end{cases}$$

# Random numbers - Continuous

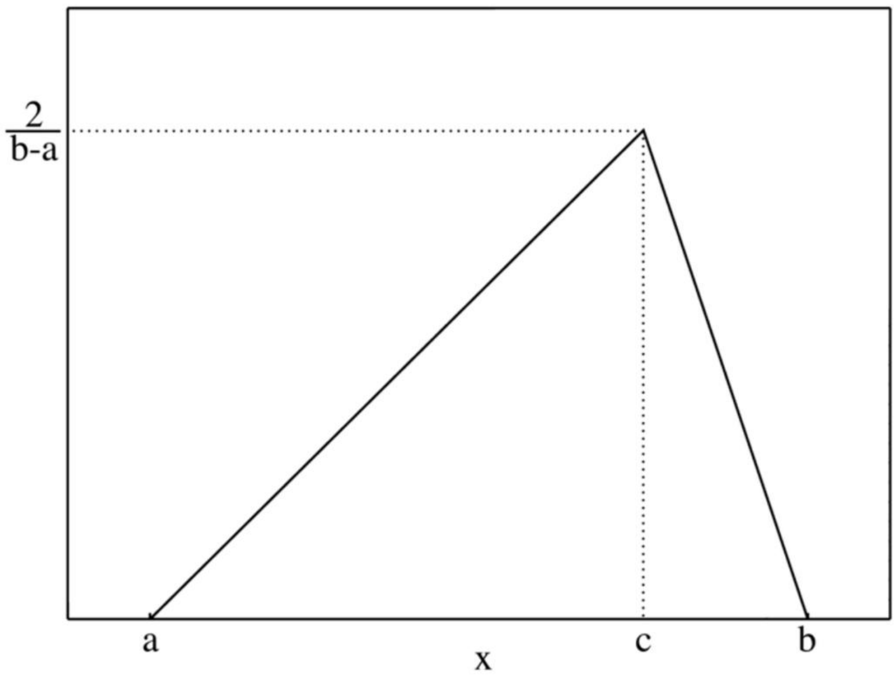
- **Triangle distribution (2/3):** $RV \quad X \sim triang(a,b,c)$ (LK 8.3.15)

  - Mode c
  - Range $[a;b]$
  - Expectation: $E(X) = \dfrac{a+b}{2}$
  - Variance: $VAR(X) = \dfrac{(a^2+b^2+c^2-ab-ac-bc)}{12}$
  - Generation: Inversion (U < c)

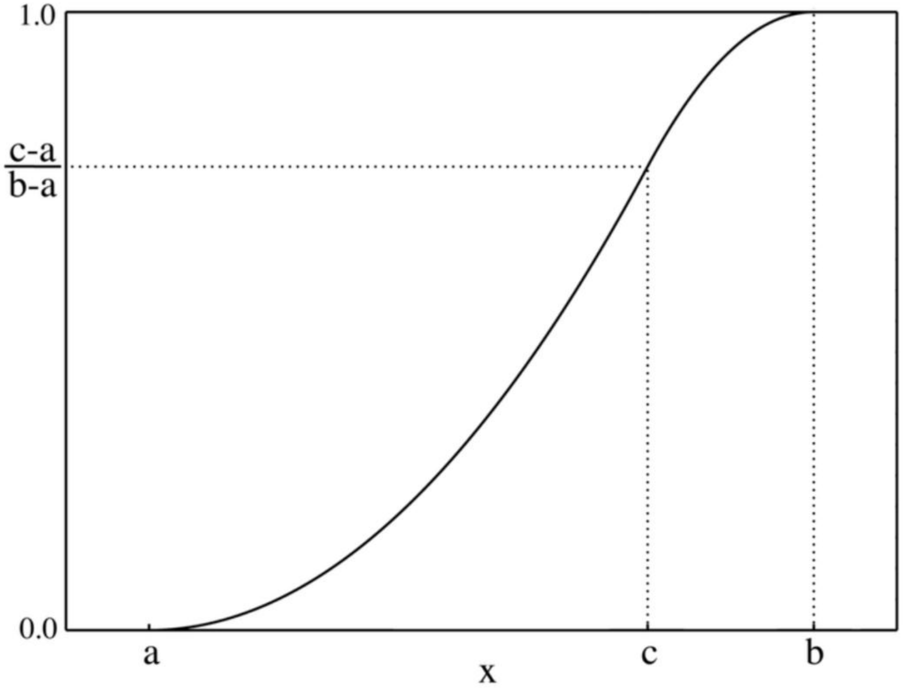
    $U \sim U(0,1), X \sim triang(0,1,c) \qquad 0 < c < 1$

# Random numbers - Continuous

- **Triangle distribution (3/3):** $RV\ \ X \sim triang(a,b,c)$ (LK 8.3.15)




Probability Density Function




Cumulative Density Function

Pictures taken from Wikipedia

# Random numbers - Continuous

- **Normal distribution(1/3):** $RV\ \ X \sim N(\mu,\sigma^2)$ (LK 8.3.6)
  - Density function: $f(x) = \dfrac{1}{\sqrt{2\cdot\pi\cdot\sigma^2}}\cdot e^{-\left(\frac{(x-\mu)^2}{2\cdot\sigma^2}\right)}$
  - Distribution function: $F(x) = \dfrac{1}{\sigma\sqrt{2\pi}}\int\limits_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt$
  - Range: $\left]-\infty;\infty\right[$
  - Mode: $\mu$
  - Expectation: $E(X)=\mu$
  - Variance: $VAR(X)=\sigma^2$
  - Scalability: $X \sim N(0,1) \Rightarrow (\mu+\sigma X) \sim N(\mu,\sigma^2)$

# Random numbers - Continuous

- **Normal distribution(2/3):** $RV\ \ X \sim N(\mu,\sigma^2)$ (LK 8.3.6)
  - Generation Accept-Reject
    - Two independent random variables $U_1, U_2 \sim U(0,1)$
    - $V_i = 2U_i - 1$
    - $W = V_1^2 + V_2^2$
    - Algorithm:

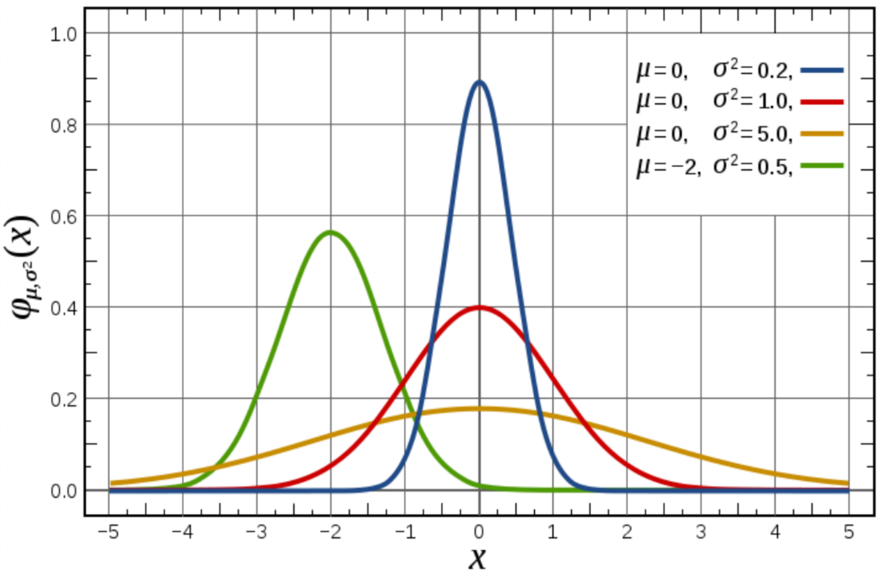
Accept if $W \leq 1$

$$Y = \sqrt{\frac{-2\ln W}{W}} \quad , \quad X_1 = V_1 \cdot Y \quad , \quad X_2 = V_2 \cdot Y$$

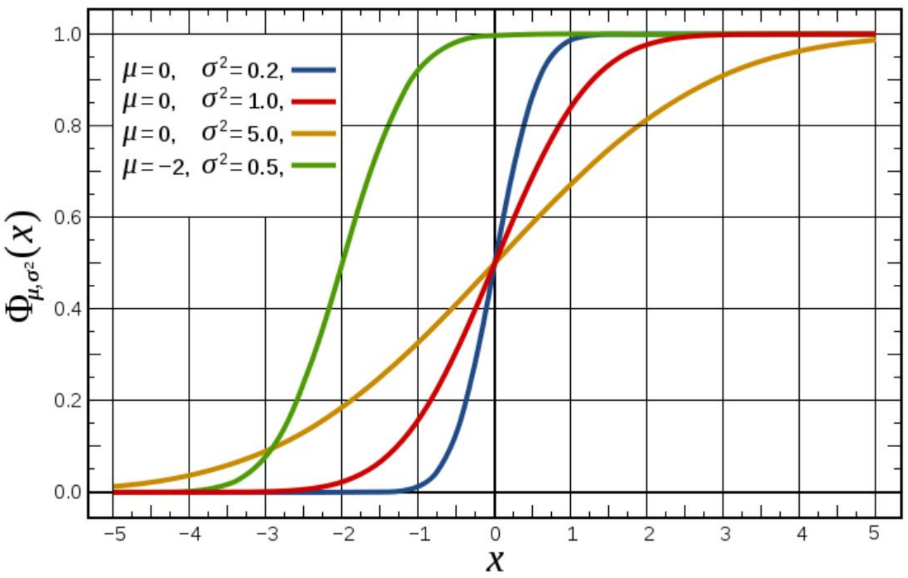
Reject otherwise

# Random numbers

- **Normal distribution(3/3):** $RV\ \ X \sim N(\mu, \sigma^2)$ (LK 8.3.6)




Probability Density Function




Cumulative Density Function

Pictures taken from Wikipedia

# Random numbers

- **Lognormal distribution(1/3):** $RV \;\; X \sim LN(\mu, \sigma^2)$ (LK 8.3.7)

  - Special property of the lognormal distribution

    if $Y \sim N(\mu, \sigma^2) \Longrightarrow e^Y \sim LN(\mu, \sigma^2)$

  - Range: $[0, \infty)$

  - Algorithm: Composition

    – $Y \sim N(\mu, \sigma^2) \Longrightarrow X = e^Y$

  - Expectation: $E(X) = e^{\mu + \frac{\sigma^2}{2}}$

  - Variance: $VAR(X) = e^{2\mu + \sigma^2}\left(e^{\sigma^2} - 1\right)$

Note that μ and σ are NOT the mean and the variance of the lognormal distribution!

# Random numbers

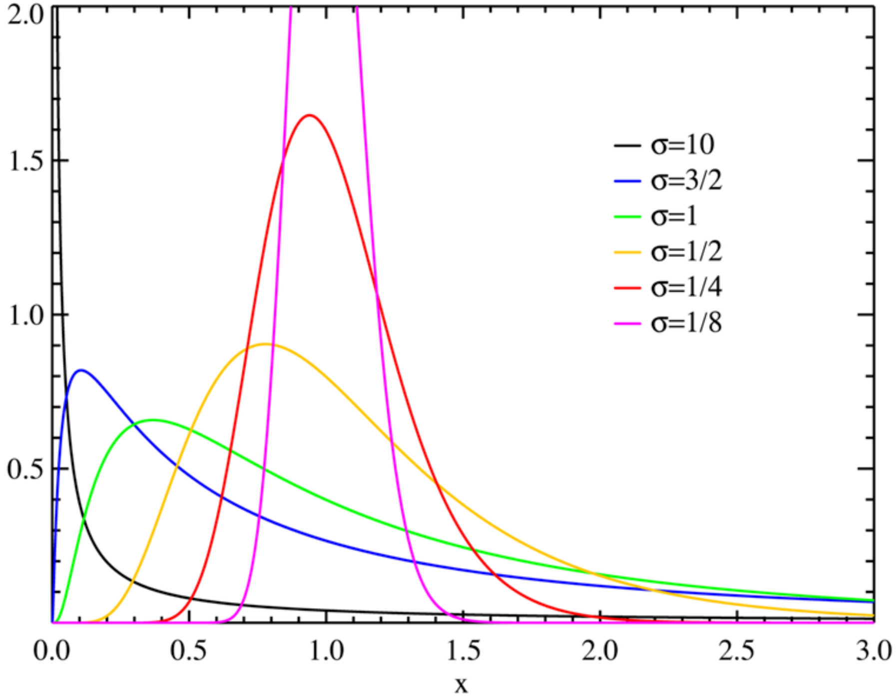
- **Lognormal distribution(2/3):** $RV\ \ X \sim LN(\mu,\sigma^2)$ (LK 8.3.7)

  - Parameters of the normal distribution which is used to generate LN

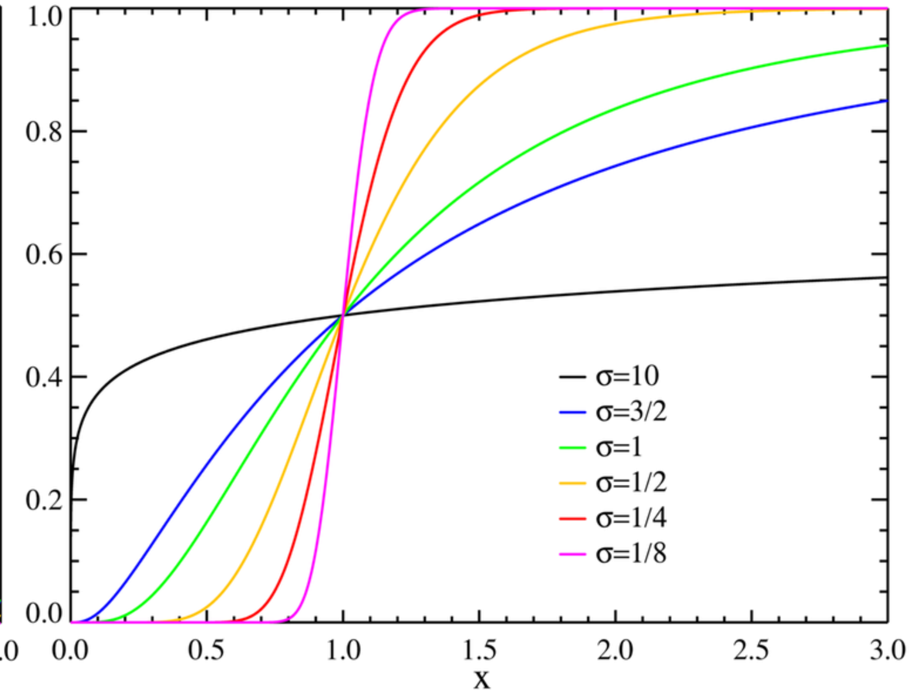
    – $\mu = E[Y] = \ln\left(\dfrac{E[X]^2}{\sqrt{E[X]^2 + VAR[X]}}\right)$

    – $\sigma^2 = VAR[Y] = \ln\left(\dfrac{E[X]^2}{\sqrt{E[X]^2 + VAR[X]}}\right)$

- **Lognormal distribution(3/3):** $RV \ \ X \sim LN(\mu, \sigma^2)$ (LK 8.3.7)




Probability Density Function




Cumulative Density Function

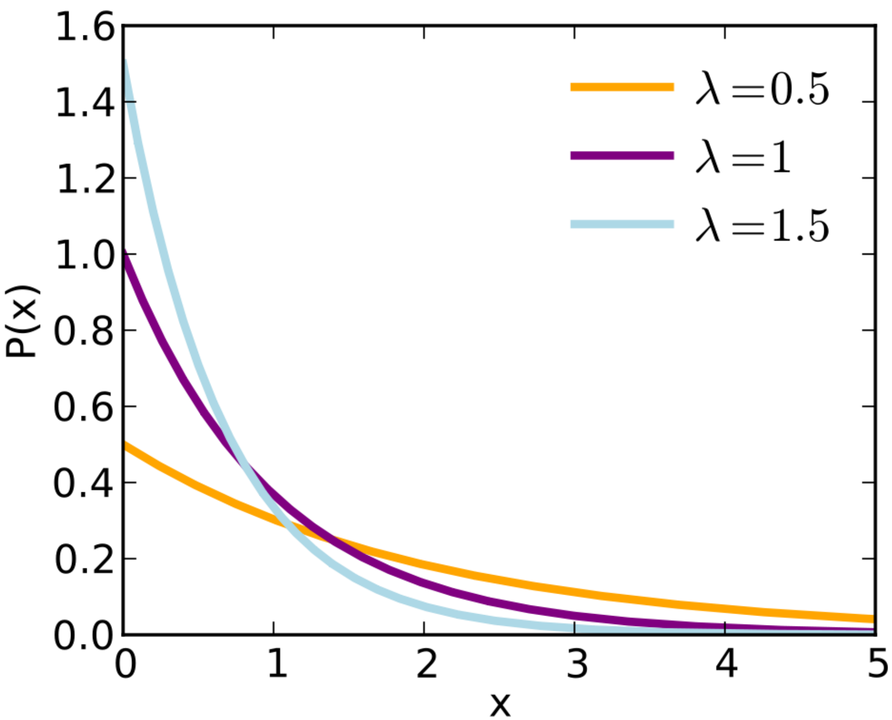
Pictures taken from Wikipedia

# Random numbers

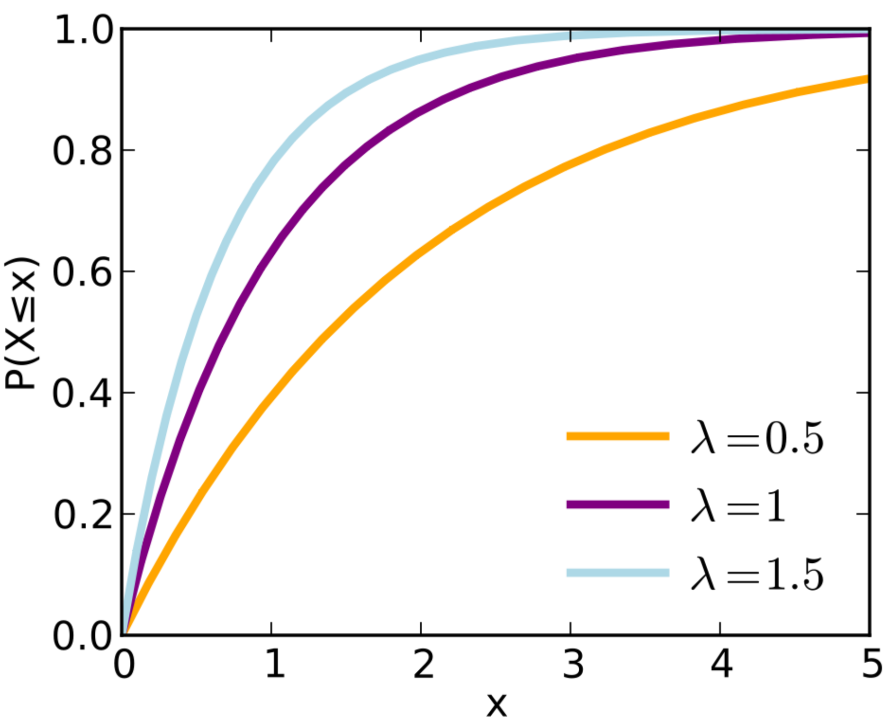
- **Exponential distribution(1/2):** $RV\ \ X \sim \exp(\lambda)$ (LK 8.3.2)
  - Density function: $f(x) = \lambda \cdot e^{-\lambda x} \quad \text{für} \quad x \geq 0$
  - Distribution function: $F(x) = 1 - e^{-\lambda x}$
  - Range: $[0, \infty[$ Mode: 0
  - Expectation: $E(X) = \dfrac{1}{\lambda}$
  - Variance: $VAR(X) = \dfrac{1}{\lambda^2}$
  - Coefficient of variation: $c_{Var} = 1$
  - Generation: Inversion $U \sim U(0,1), X = \dfrac{-\ln(U)}{\lambda}$

# Random numbers - Continuous

- Exponential distribution(2/2): $RV\ \ X \sim \exp(\lambda)$ (LK 8.3.2)




Probability Density Function




Cumulative Density Function

Pictures taken from Wikipedia

# Random numbers - Continuous

- **Erlang-k distribution(1/3):** $RV\ \ X \sim k - Erlang(\lambda)$ (LK 8.3.3)
  - $RV\ \ X = Y_1 + Y_2 + Y_3 + \cdots + Y_k$ where the Yi's are IID exponential random variables
  - Density function: $$f(x) = \begin{cases} \dfrac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} & for \quad x \geq 0 \\ 0 & Otherwise \end{cases}$$
  - Distribution function: $$F(x) = \begin{cases} 1 - e^{-\lambda x} \cdot \sum_{i=0}^{k-1} \dfrac{(\lambda x)^i}{i!} & for \quad x \geq 0 \\ 0 & Otherwise \end{cases}$$

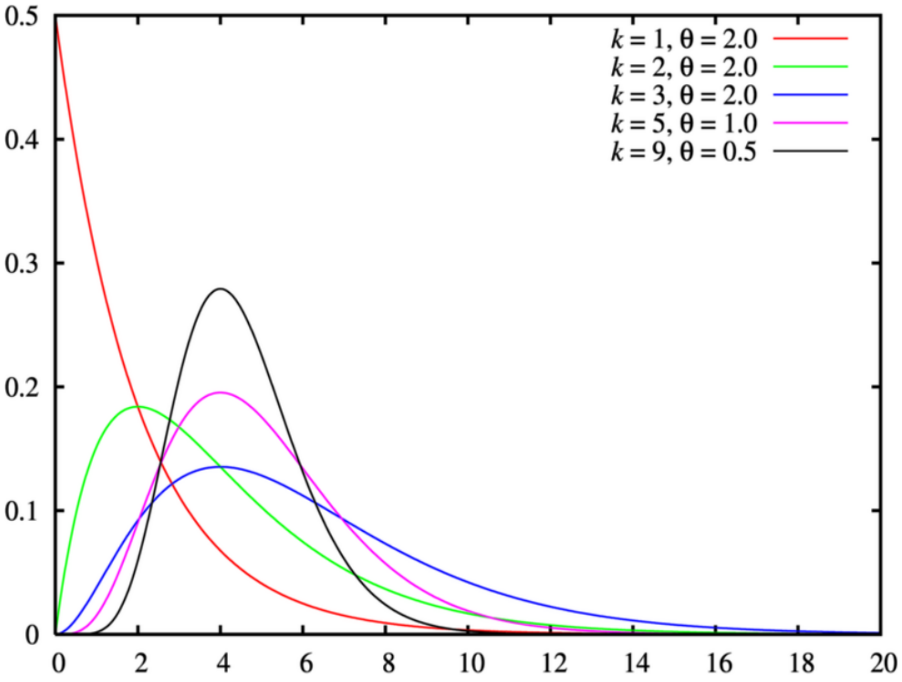
RV X represents the sum of k exponential random variables

# Random numbers - Continuous

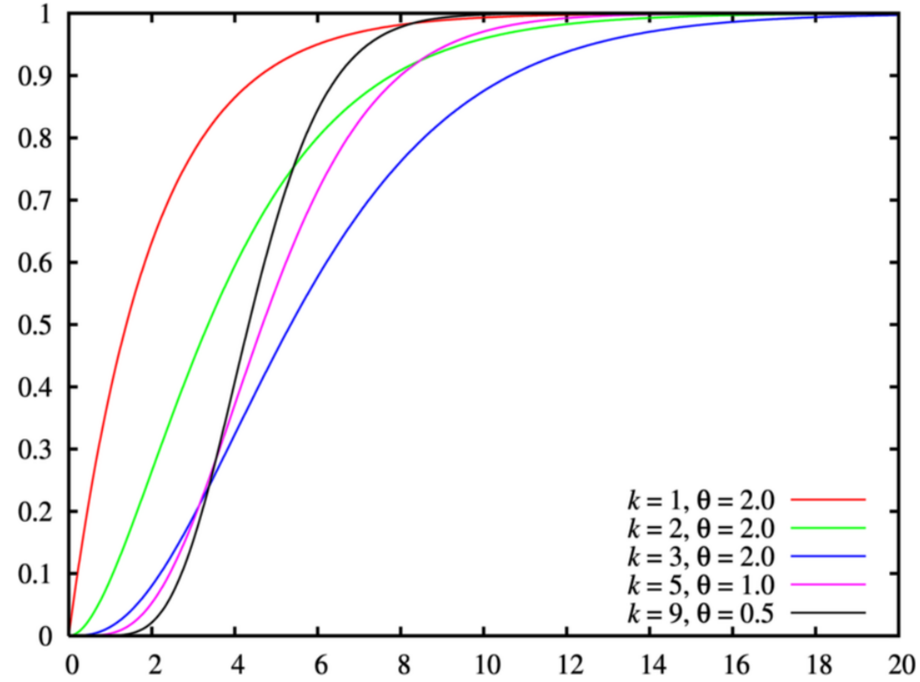
- **Erlang-k distribution(2/3):** $RV\ \ X \sim k-Erlang(\lambda)$ (LK 8.3.3)
  - Range: $[0,\infty[$
  - Expectation: $E(X) = \dfrac{k}{\lambda}$
  - Variance: $VAR(X) = \dfrac{k}{\lambda^2}$
  - Mode: $\dfrac{k-1}{\lambda}$
  - Coefficient of variation: $c_{Var} = \dfrac{1}{\sqrt{k}}$
  - Generation:
    » Inversion $U_i \sim U(0{,}1),\ X = \dfrac{-\ln\left(\prod_{0\le i<k} U_i\right)}{\lambda}$

    » Convolution $RV\ \ X = Y_1 + Y_2 + Y_3 + \cdots + Y_k$

- **Erlang-k distribution(3/3):** $RV \;\; X \sim k - Erlang(\lambda)$ (LK 8.3.3)




Probability Density Function




Cumulative Density Function

Pictures taken from Wikipedia

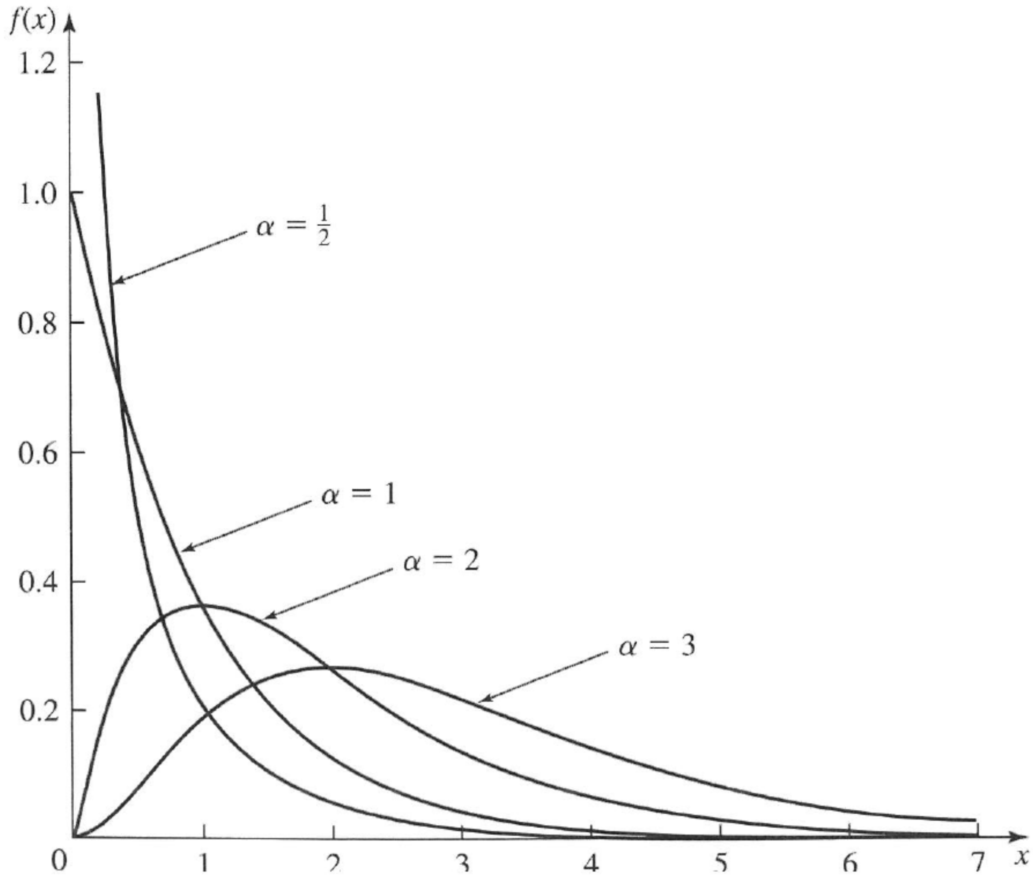
# Random numbers - Continuous

- **Gamma distribution(1/3):** $RV\ \ X \sim gamma(\alpha, \beta)$ (LK 8.3.4)
  - Density function: $$f(x) = \begin{cases} \dfrac{\beta^{-\alpha} \cdot x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha)} & for \quad x \le 0 \\ 0 & Otherwise \end{cases}$$
  - Distribution function: $$F(x) = \begin{cases} 1 - e^{\frac{-x}{\beta}} \cdot \sum\limits_{0 \le i < \alpha} \dfrac{\left(\frac{-x}{\beta}\right)^i}{i!} & for \quad x > 0 \\ 0 & Otherwise \end{cases}$$
  - Parameter description:
    - Location parameter γ: Shifting the distribution along the x-axis
    - Scale parameter β: Linear impact on the expectation
    - Shape parameter α: Changes the shape of the distribution

# Random numbers - Continuous

- **Gamma distribution(2/3):** $RV\ \ X \sim gamma(\alpha, \beta)$ (LK 8.3.4)
  - Gamma function: $$\Gamma(z) = \begin{cases} \int_0^\infty t^{z-1} e^{-t} dt & if \ \ x \geq 0 \\ 0 & if \ \ x < 0 \end{cases}$$
  - Expectation: $E(X) = \alpha \cdot \beta$
  - Coefficient of variation: $c_{Var} = 1$
  - Mode: $$\begin{cases} 0 & if \ \ \alpha < 1 \\ \beta \cdot (\alpha - 1) & if \ \ \alpha \geq 1 \end{cases}$$
  - Generation:
    - Step 1 $X \sim gamma(\alpha, \beta) \rightarrow X = \beta \cdot Y \qquad Y \sim gamma(\alpha, 1)$
    - Step 2 Generation of $X \sim gamma(\alpha, 1)$ with Accept-Reject

- **Gamma distribution(3/3):** $RV\ \ X \sim gamma(\alpha, \beta)$ (LK 8.3.4)




Probability Density Function

Picture taken from LK, p. 285

# Random numbers - Discrete

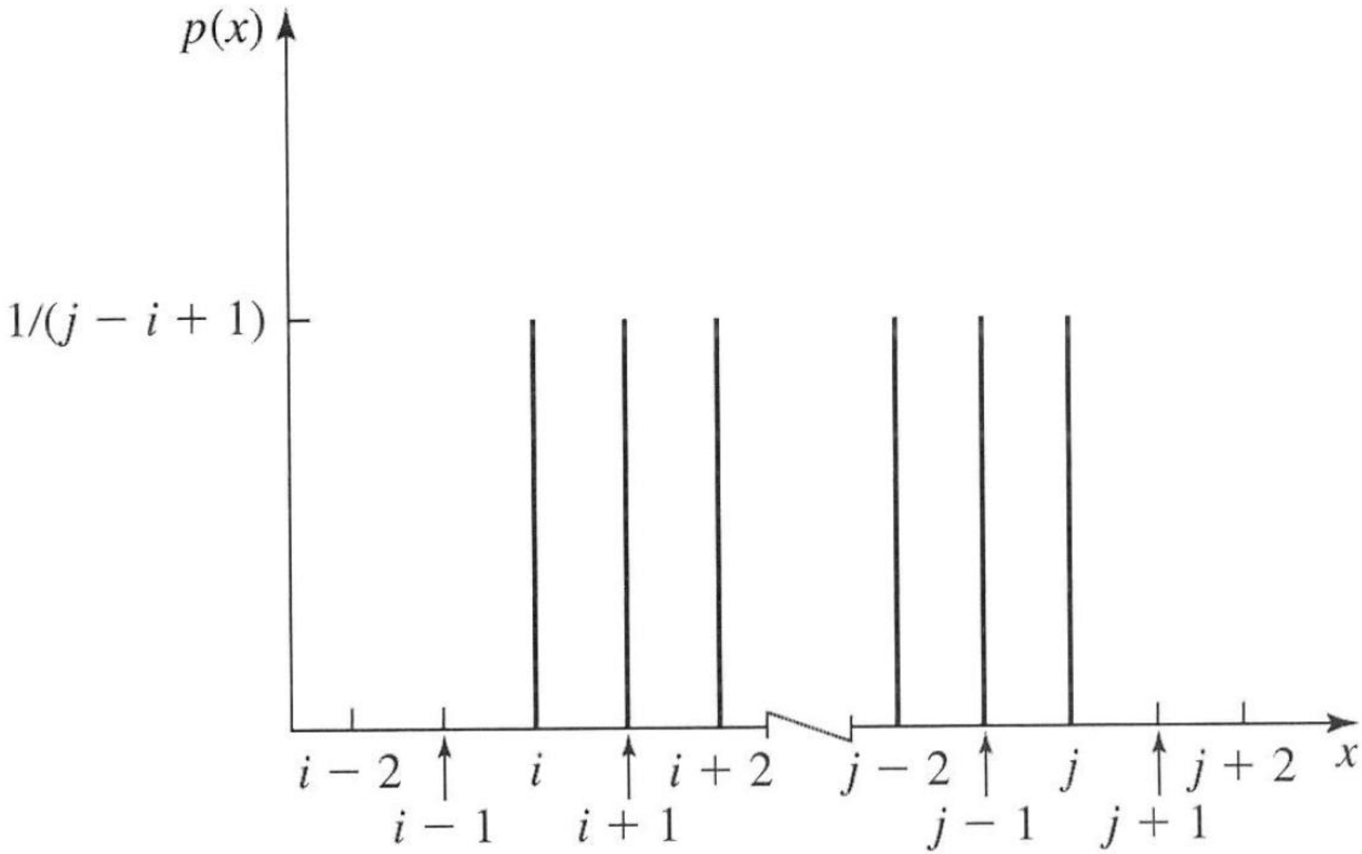
- **Uniform (discrete) (1/2)** $RV\ \ X \sim DU(i,j)$ (LK 8.4.2)
  - Distribution: $$p(k) = \begin{cases} \dfrac{1}{j-i+1} & if\ \ k \in \{i, i+1, i+2, \ldots, j\} \\ 0 & Otherwise \end{cases}$$
  - Range: $i \le k \le j$
  - Expectation: $$E(X) = \frac{(i+j)}{2}$$
  - Variance: $$VAR(X) = \frac{(j-i+1)^2 - 1}{12}$$
  - Generation: Inversion

    $U \sim U(0,1) \qquad X = i + \lfloor (j-i+1) \cdot U \rfloor$

DU(0,1) and Bernoulli(0.5) distributions are the same

# Random numbers - Discrete

- **Uniform (discrete) (2/2)** $RV\ \ X \sim DU(i, j)$ (LK 8.4.2)




Distribution

- **Bernoulli (1/2)** $RV \;\; X \sim Bernoulli\ (p)$ (LK 8.4.1)
  - Example: Flipping a coin 
  - Distribution: $$p(k)=\begin{cases}1-p & if\;\; k=0\\ p & if\;\; k=1\\ 0 & Otherwise\end{cases}$$
  - Range: $$i \le k \le j$$
  - Expectation: $$E(X)=p$$
  - Variance: $$VAR(X)=p\cdot(1-p)$$
  - Coefficient of variation: $$c_{Var}=\sqrt{\frac{1-p}{n\cdot p}}$$

# Random numbers - Discrete

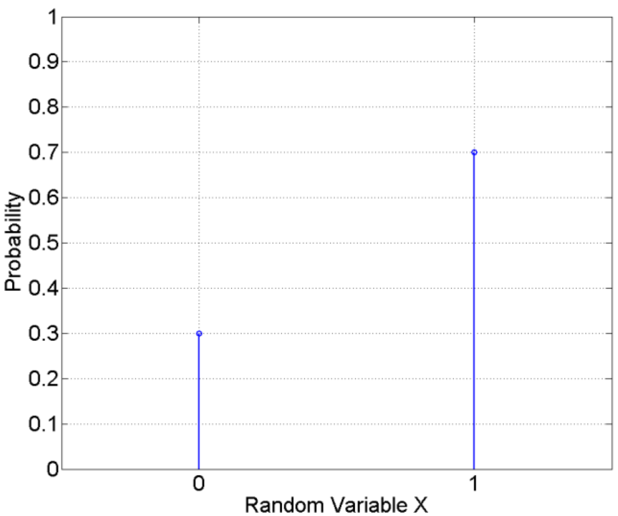
- **Bernoulli (2/2)** $RV\ \ X \sim Bernoulli\ \ (p)$ (LK 8.4.1)

  - Mode: 0 or 1 (depends on the definition of the outcome)
  - Generation: Inversion $U \sim U(0,1)$

$$X = \begin{cases} 0 \ \ if \ \ U < p \\ 1 \ \ Otherwise \end{cases}$$

  - Distribution

$Bernoulli\ \ (0.3)$



# Random numbers - Discrete

- **N-Bernoulli (1/2)** $RV\ \ X \sim Bernoulli\ (n, p)$ (LK 8.4.4)

  - Example: Flipping a coin n times

  - Distribution: $p(k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \qquad 0 \le k \le n$
  - Range: $0 \le k \le n$
  - Expectation: $E(X) = np$
  - Variance: $VAR(X) = n \cdot p \cdot (1-p)$
  - Coefficient of variation: $c_{Var} = \sqrt{\dfrac{1-p}{n \cdot p}}$

# Random numbers - Discrete

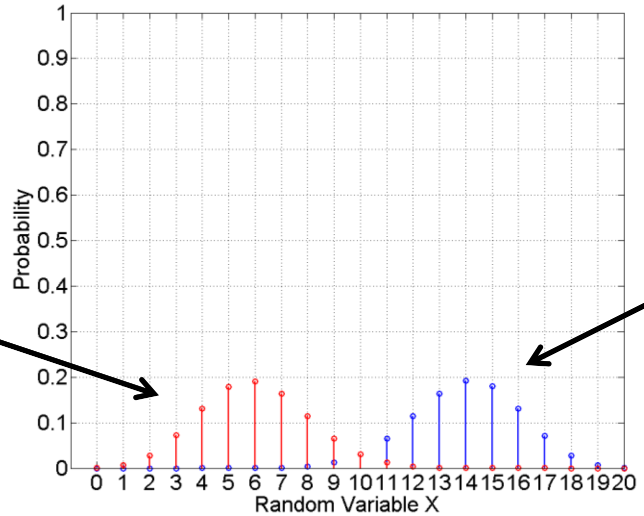
- **N-Bernoulli (2/2)** $RV \;\; X \sim Bernoulli \;\; (n, p)$ (LK 8.4.4)

  - Mode: 0 or 1 (depends on the definition of the outcome)
  - Generation: Composition

$$Bernoulli \;\; (n, p) \approx \sum_{0 \le i < n} Bernoulli \;\; (p)$$

  - Distribution

$Bernoulli \;\; (20{,}0.3)$




$Bernoulli \;\; (20{,}0.7)$

# Random numbers - Discrete

- **Geom (1/2)** $RV \;\; X \sim Geom\,(p)$ (LK 8.4.5)

  - Example: Number of unsuccessful Bernoulli – Experiments until a successful outcome (e.g. number of retransmissions)
  - Distribution: $p(x) = p \cdot (1-p)^x$
  - Distribution function: $F(x) = 1-(1-p)^{\lfloor x \rfloor + 1}$
  - Expectation: $E(X) = \dfrac{1-p}{p}$
  - Variance: $VAR(X) = \dfrac{1-p}{p^2}$
  - Coefficient of variation: $c_{Var} = \sqrt{\dfrac{1}{1-p}}$

# Random numbers - Discrete

- **Geom (2/2)** $RV \;\; X \sim Geom\,(p)$ (LK 8.4.5)

  - Mode: 0

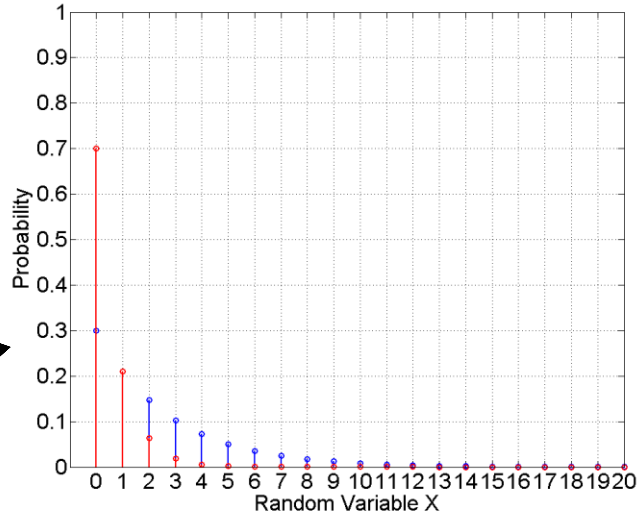
  - Generation: Inversion $U \sim U(0,1)$

$$X = \left\lfloor \frac{\ln(U)}{\ln(1-p)} \right\rfloor$$

  - Distribution

$Geom\,(0.7)$

$Geom\,(0.3)$




$$\boxed{p(0) = p}$$

# Random numbers - Discrete

## Poisson(1/3) $RV\ \ X \sim Poisson\ (\lambda)$ (LK 6.2.4)

- Example: Number of events that occur in an interval of time when the events are occurring at a constant rate (number of items in a batch of random size)
- Distribution: $$p(x) = \frac{\lambda^x}{x!} \cdot e^{-\lambda} \qquad if\ \ x \in \{0,1,2,\ldots\}$$
- Distribution function: $$F(x) = \begin{cases} e^{-\lambda} \sum_{i=0}^{\lfloor x \rfloor} \frac{\lambda^i}{i!} & if\ \ x \geq 0 \\ 0 & if\ \ x < 0 \end{cases}$$
- Parameter: $\lambda > 0$

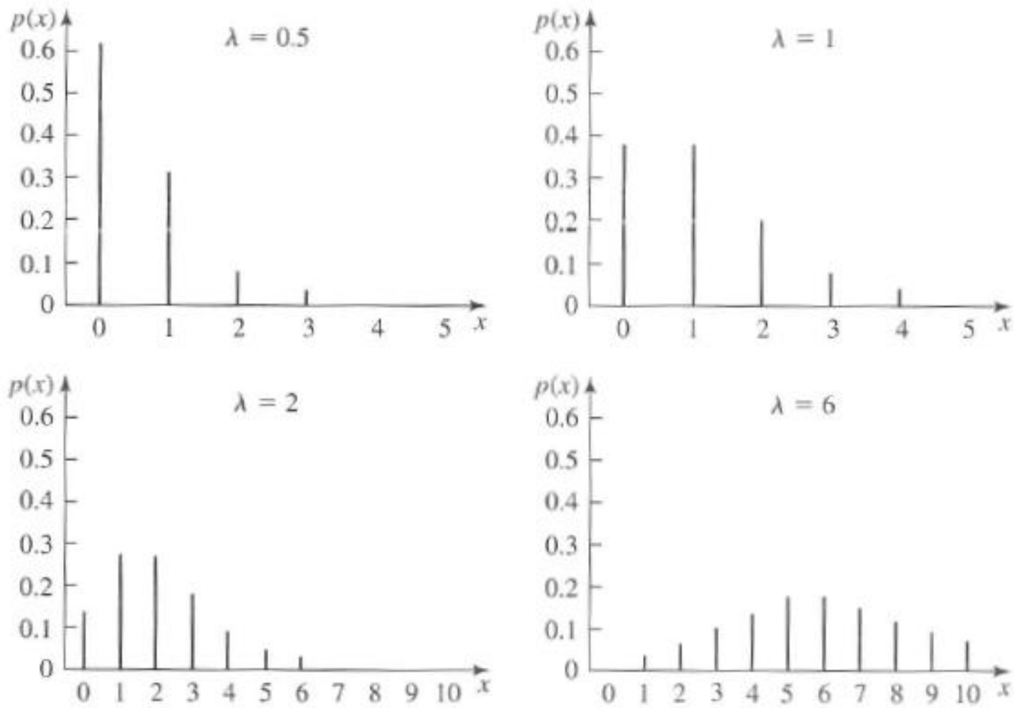
# Random numbers - Discrete

- **Poisson(2/3)** $RV\ \ X \sim Poisson\ (\lambda)$ (LK 6.2.4)

  - Range: $\{0,1,2,3,\ldots\}$
  - Expectation: $E(X)=\lambda$
  - Variance: $VAR(X)=\lambda$
  - Coefficient of variation: $c_{Var}=\frac{1}{\sqrt{\lambda}}$
  - Mode $\begin{cases}\lambda \cap \lambda-1 & \text{λ is an integer}\\ \lfloor\lambda\rfloor & \text{otherwise}\end{cases}$
  - Special characteristics:
    - $x=0$ ⇨ exponential distribution (time interval between two consecutive events)
    - Number of events until a certain point in time is Poisson distributed
    - Period of time until n events have occurred is Erlang distributed

# Random numbers - Discrete

- **Poisson(3/3)** $RV \;\; X \sim Poisson\ (\lambda)$ (LK 6.2.4)




Picture taken from LK, p.309

# Random numbers - Discrete

- **General Discrete(1/1)** $RV \;\; X \sim GD$ (LK 8.4.3)

  - Distribution:

$$p(x) = \begin{cases} p_k & if \;\; x = x_k, \;\; 0 \le k < n \\ 0 & Otherwise \end{cases}$$

  - Generation: Inversion $U \sim U(0,1)$

$X = x_k$ , falls $\sum_{j=0}^{k-1} p_j \le U < \sum_{j=0}^{k} p_j$

# Random number generator algorithms and their quality

Some slides/figures taken from:
Oliver Rose
Averill Law, David Kelton
Wikimedia Commons (user Matt Crypto)
Dilbert

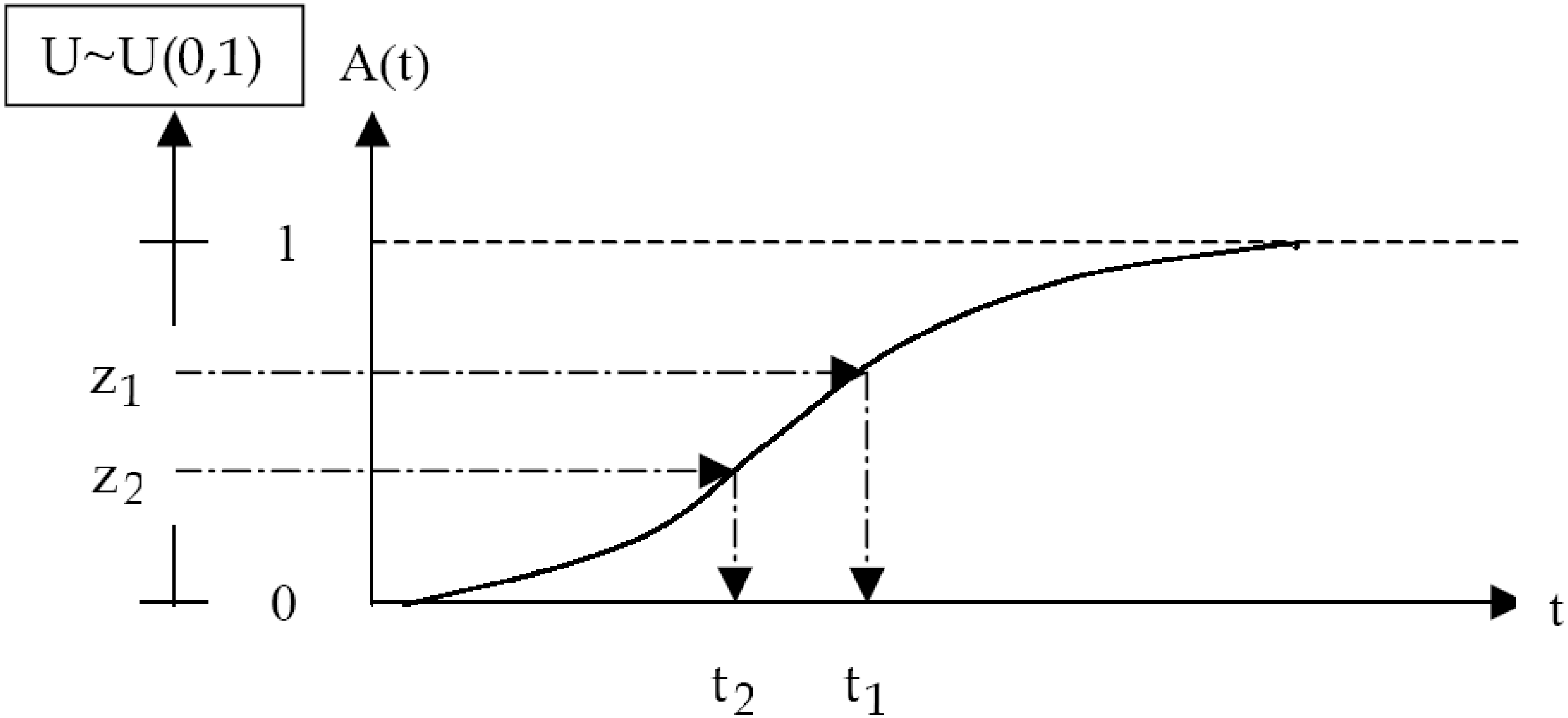
# Topics

- Random Number Generator (RNG)
- Linear Congruential Generator (LCG)
- $X^2$ Test
- Serial Test
- Spectral Test
- Shift Register
- Generalised Feedback Shift Register
- Mersenne Twister

# Structure of this lecture

- Generating U(0,1) random numbers
  - Motivation
  - Overview on RNG families
- Linear Congruential generators (LCG)
- Statistical properties, statistical (empirical) tests
  - $\chi^2$ test for uniformity
  - Correlation tests: Runs-up, sequence
- Theoretical aspects, theoretical tests
  - Period length
  - Spectral test
- RNG that are better than LCG

# Recall the inversion method




- Generate uniformly distributed numbers ∈ 0.0 … 1.0
- Compute inverse $A^{-1}(t)$ of PDF A(t)
- Generate samples

# Generating U(0,1) random numbers is crucial

- For all random number generation methods, we need uniformly distributed random numbers from ]0,1[
  $\Rightarrow$ U(0,1) random numbers are required
- Mandatory characteristics
  - Random (…obviously)
  - Uniform (make use of the whole distribution function)
  - Uncorrelated (no dependencies): difficult!
  - Reproducible (for verification of experiments)
    $\rightarrow$ use pseudo random numbers
  - Fast (usually, there is a need for a lot of samples)

# RNG in simulation vs. RNG in cryptography

- Also need for random numbers in cryptography
  - Key generation
  - Challenge generation in challenge-response systems
  - …
- Additional requirement:
  - Prediction of future “random” values by sampling previous values must not be possible
  - In simulation: not an issue if there is no real correlation
- Lighter requirement:
  - RNs are not used constantly, only in ~start-up phases
    $\Rightarrow$ speed is not of much importance
  - In simulation: need lots of numbers
    $\Rightarrow$ speed is very important

# Generation of U(0,1) random numbers: overview

Main families:

- Linear Congruential Generator (LCG): the simplest
- General Congruential Generators
  - Quadratic Congruential Generator
  - Multiple recursive generators
- Shift register with feedback (Tausworthe)
  - E.g., Mersenne Twister: state-of-the-art
- Composite generators: output of multiple RNG
  - E.g., use one to shuffle (“twist”) the output of the other

# RNG: alternatives unsuitable for simulation

- Algorithms from cryptography
  - For example: counter→AES, counter→SHA1, counter→MD5, etc.
  - Usually way too slow
- Calculate transcendent numbers (e.g., π or e), view their digits as random
  - E.g.: digits of 100,000$^{th}$ decimal place of π onwards
  - Problem: Are they really random? There seems to be some structure…
- Physical generators (cf. previous lecture)
  - Not reproducible, no seed
- Tables with pre-computed random numbers
  - We need too many random numbers, the tables would have to be huge…

# Linear Congruential Generators

- Calculate RN from previous RN using some formula
- Sequence of integers $Z_1, Z_2, \ldots$ defined by

$$Z_i = (a \cdot Z_{i-1} + c) \pmod{m}$$

- with modulus $m$, multiplier $a$,

  increment $c$, and seed $Z_0$

- $c$=0: multiplicative LCG

  Example:

$$Z_i = 16807 \cdot Z_{i-1} \quad \left(\bmod 2^{31} - 1\right)$$

(Lewis, Goodman, Miller, 1969)

- $c$>0: mixed LCG

# …but they don't create floats, but integers > 1?!

- Obviously,

  $Z_i$ = something mod m

  and

  something mod m < m

- ⇒ Just normalise the result!
  - Divide by m? But then, 1.0 cannot be attained.
  - Better: Divide by m–1.

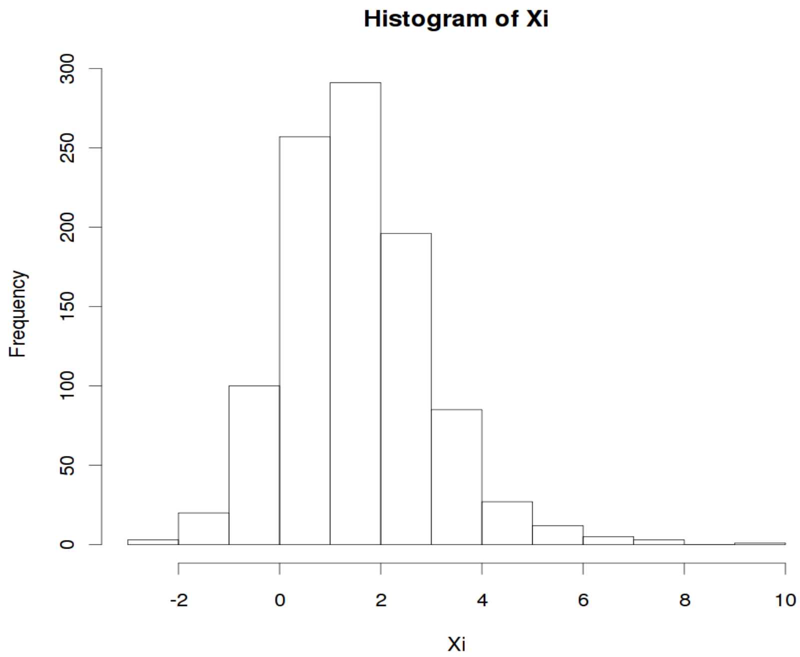
# Do they really generate uniformly distributed random numbers?

- Test for uniformity:
  - Create a number of samples from RNG
  - Test if these numbers are uniformly distributed
- A number of statistical tests to do this:
  - $\chi^2$ test (deutsch: Chi-Quadrat-Anpassungstest)
  - Kolmogorov-Smirnov test
  - … and a whole lot of others! For example:
    - Cramér-von Mises test
    - Anderson-Darling test
- Graphical examination (not real tests):
  - Plot histogram / density / PDF
  - Distribution-function-difference plot
  - Quantile-quantile plot (Q-Q plot)
  - Probability-probability plot (P-P plot)

(later in course)

# Histogram

- Given a series of n measurements $X_i$
- Partition the domain $\min\{X_i\} \ldots \max\{X_i\}$ into m intervals $I_1 \ldots I_m$
- Count how many $X_i$ fall into which interval $I_j$
- Plot it:

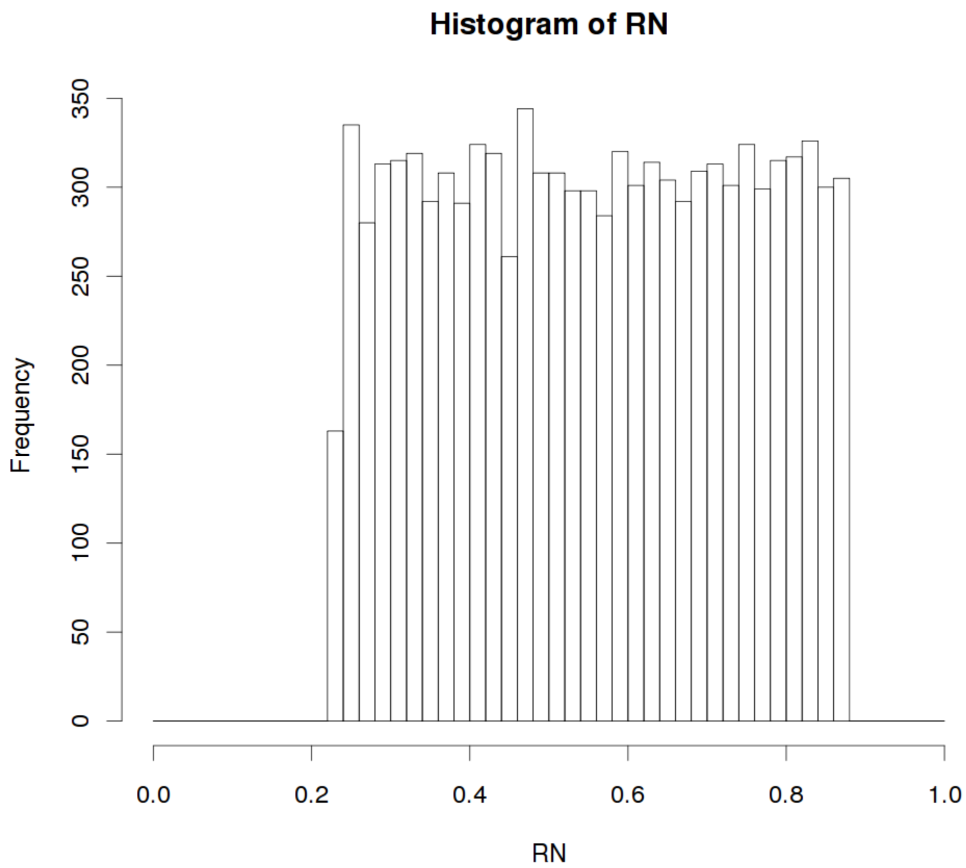



- ~discretised density function
- Recommendation:

$$m \approx \sqrt{n}$$

# What the histogram can reveal (1)

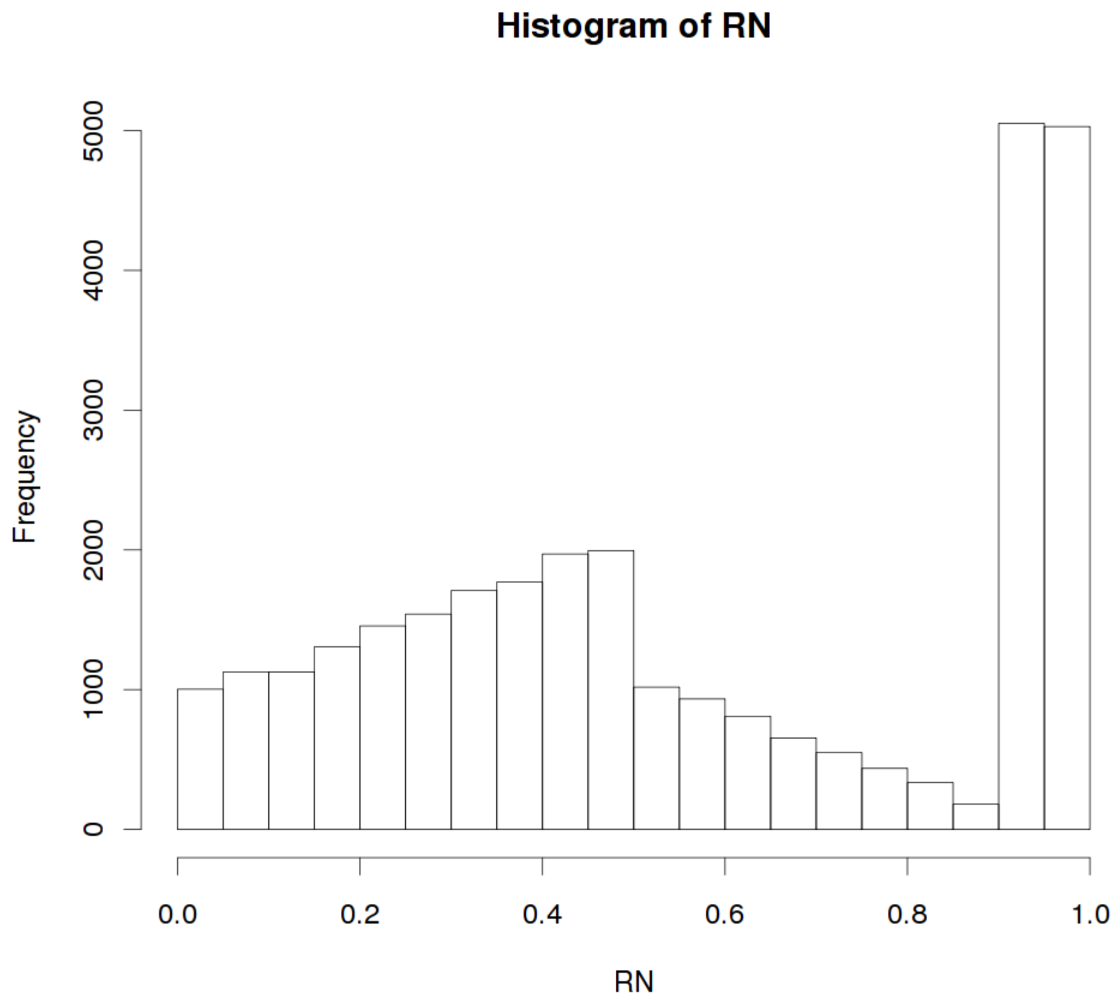
Obviously not U(0,1) random variables:




(…okay,

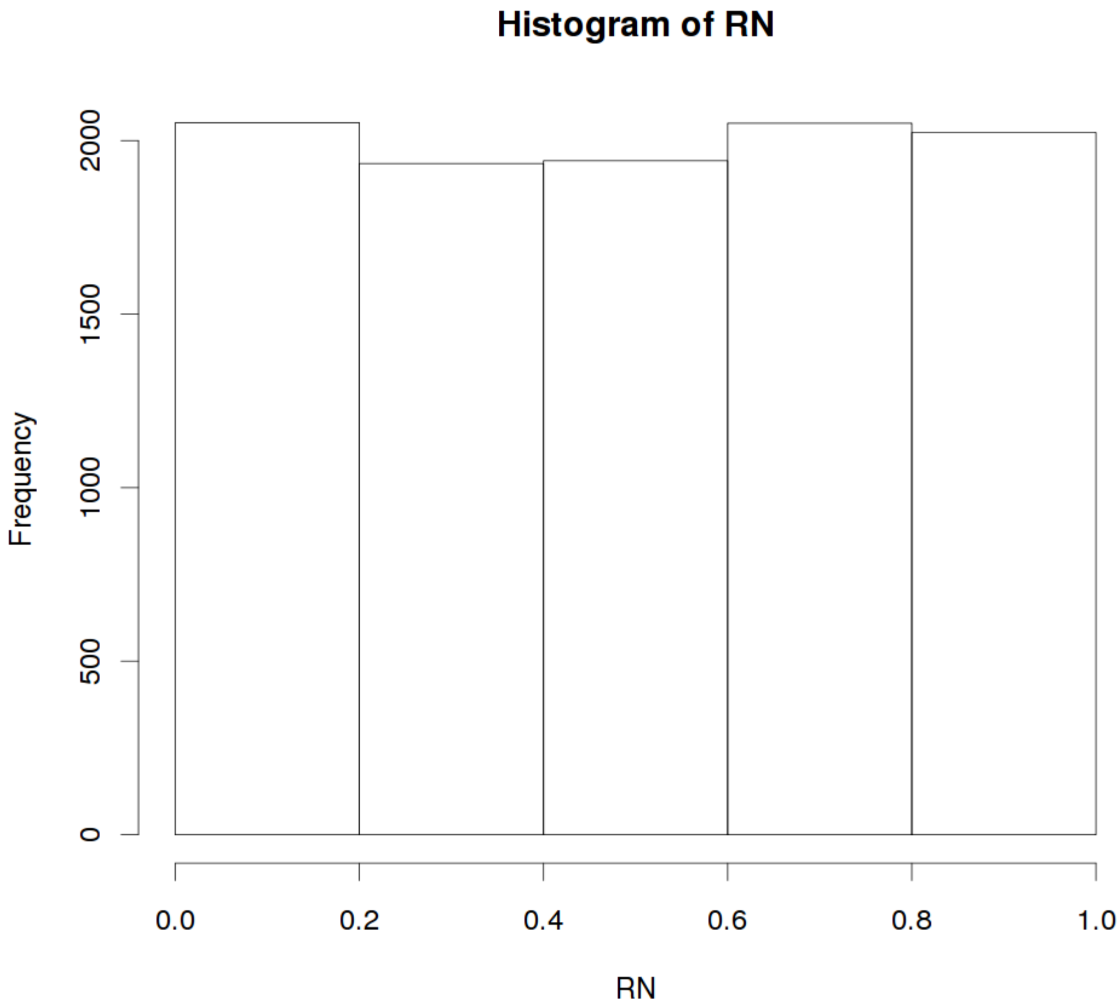
# What the histogram can reveal (2)

Obviously not U(0,1) random variables:



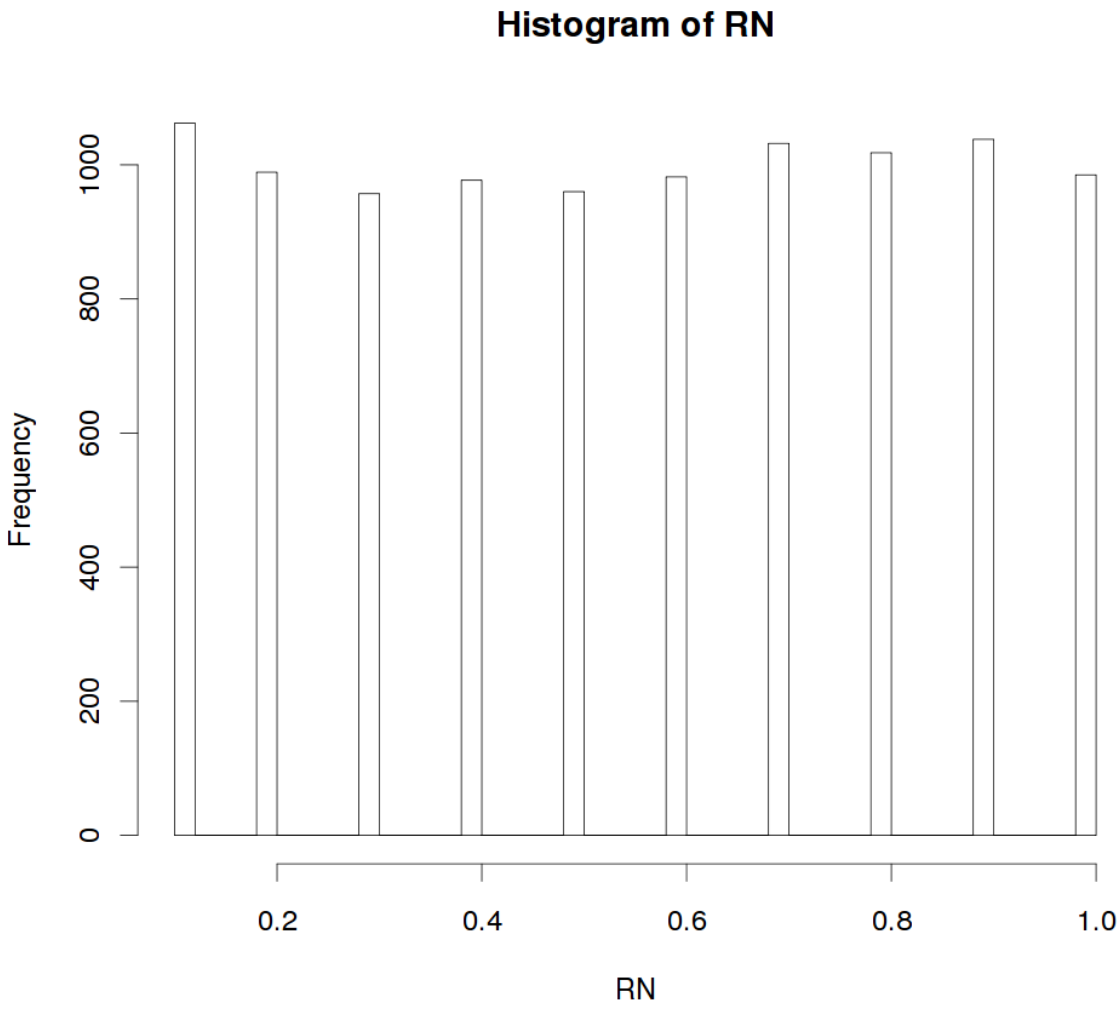
# What the histogram can reveal (3a)

Looks like a U(0,1) random variable…:



# What the histogram can reveal (3b)

…but obviously not U(0,1) random variables: huge gaps!



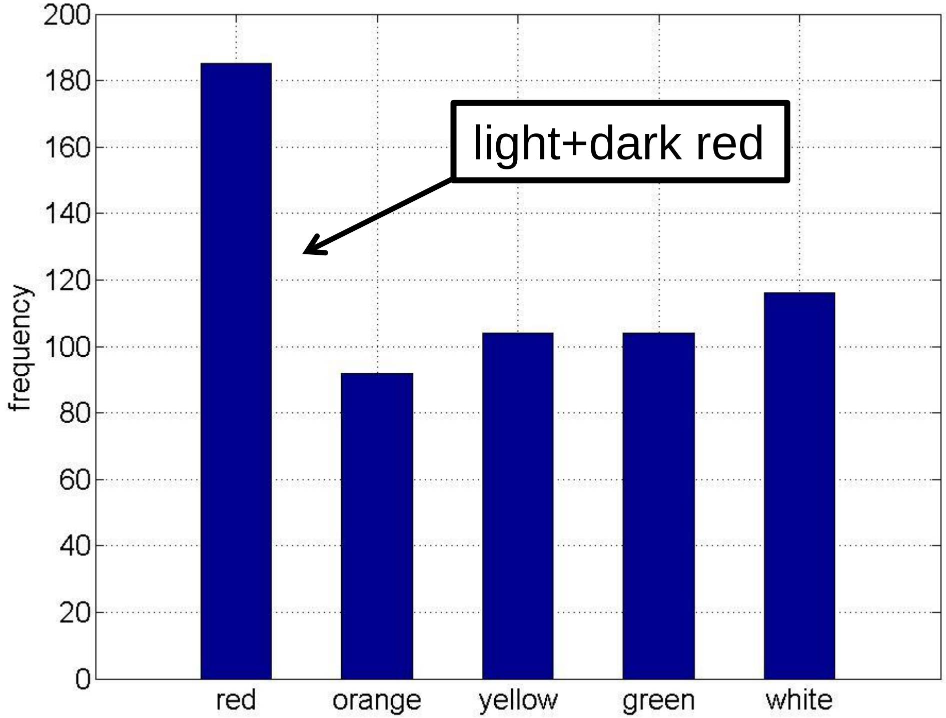
# Histograms

- Gummibears – Original - 300g – (~130 Gummibears per package)

Histograms are based on samples taken from a 300g packages

# Histograms

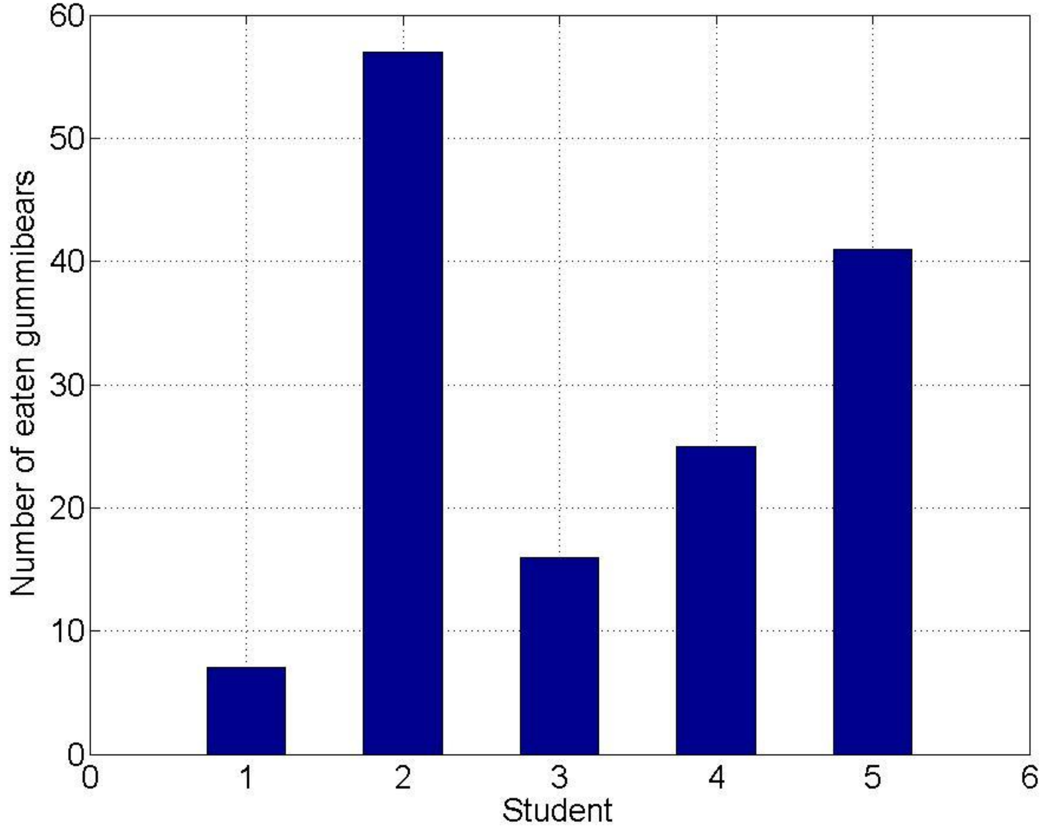
- Gummibears – Original – 1500g




Histogram is based on samples taken from 5 x 300g packages

# Histograms

- Gummibears – Eaten by students during the lecture



# Statistical tests

- Scenario: Given a set of measurements, we want to check if they conform to a distribution; here: U(0,1)
- Graphs like presented before are nice indicators,
  but not really tangible: "How straight is that line?" etc.
- We want clearer things: Numbers or yes/no decisions
- Statistical tests can do the trick, but…
  - Warning #1: Tests only can tell if measurements do not fit a particular distribution—i.e., no "yes, it fits" proof!
  - Warning #2: The result is never absolutely certain, there is always an error margin.
  - Warning #3: Usually, the input must be 'iid':
    - Independent
    - Identically distributed
  - ⇒You never get a 'proof', not even with an error margin!

# $\chi^2$ test (Pearson, 1900)

- Input:
  - Series of n measurements $X_1 \dots X_n$
  - A distribution function $f$ (the ‘theoretical function’)
- Measurements will be tested against the distribution
  - ~formal comparison of a histogram with the density function of the theoretical function
- Null hypothesis H0:
  The $X_i$ are IID random variables with distribution function $f$

# $\chi^2$ test: How it works

- Divide [0…1] into k equal-size intervals
- Count how many $X_i$ fall into which interval (histogram):
  $N_j$ := number of $X_i$ in $j$-th interval $[a_{j-1} \ldots a_j[$
- Calculate how many $X_i$ would fall into the $j$-th interval if they were sampled from the theoretical distribution:

$$p_j := \int_{a_{j-1}}^{a_j} f(x)dx$$ (*f*: density of theor. dist.)

- Calculate squared normalized difference between the observed and the expected:

$$\chi^2 := \sum_{j=1}^{k} \frac{(N_j - np_j)^2}{np_j}$$

- Obviously, if $\chi^2$ is "too large", the differences are too large, and we must reject the null hypothesis
- But what is "too large"?

# $\chi^2$ test: Using the $\chi^2$ distribution

- $\chi^2$ distribution
  - A test distribution
  - Parameter: degrees of freedom (short df)
  - $\chi^2$(k–1 df) = $\Gamma$(½(k–1) , 2) (gamma distribution)
  - Mathematically: The sum of n independent squared normal distributions
- Compare the calculated $\chi^2$ against the $\chi^2$ distribution
  - If we use k intervals, then $\chi^2$ is distributed corresponding to the $\chi^2$ distribution with k–1 df
  - Let $\chi^2_{k-1,1-\alpha}$ be the (1–α) quantile of the distribution
  - α is called the confidence level
  - Reject H0 if $\chi^2 > \chi^2_{k-1,1-\alpha}$ (i.e., the $X_i$ do not follow the theoretical distribution function)

# $\chi^2$ test and degrees of freedom

- $\chi^2$ test can be used to test against any distribution
- Easy in our case: We know the parameters of the theoretical distribution $f$ —it's U(0,1)
- Different in the general case:
  - For example, we may know it's N($\mu$, $\sigma$) (normal distribution) but we know neither $\mu$ nor $\sigma$
  - Fitting a distribution: Find parameters for $f$ that make $f$ fit the measurements $X_i$ best
  - Topic of a later lecture
- Theoretically:
  Have to estimate m parameters ⇒ Also have to take $\chi^2_{k-m-1,1-\alpha}$ into account
- Practically:
  m≤2 and large k ⇒ Don't care…

# $\chi^2$: which parameters?

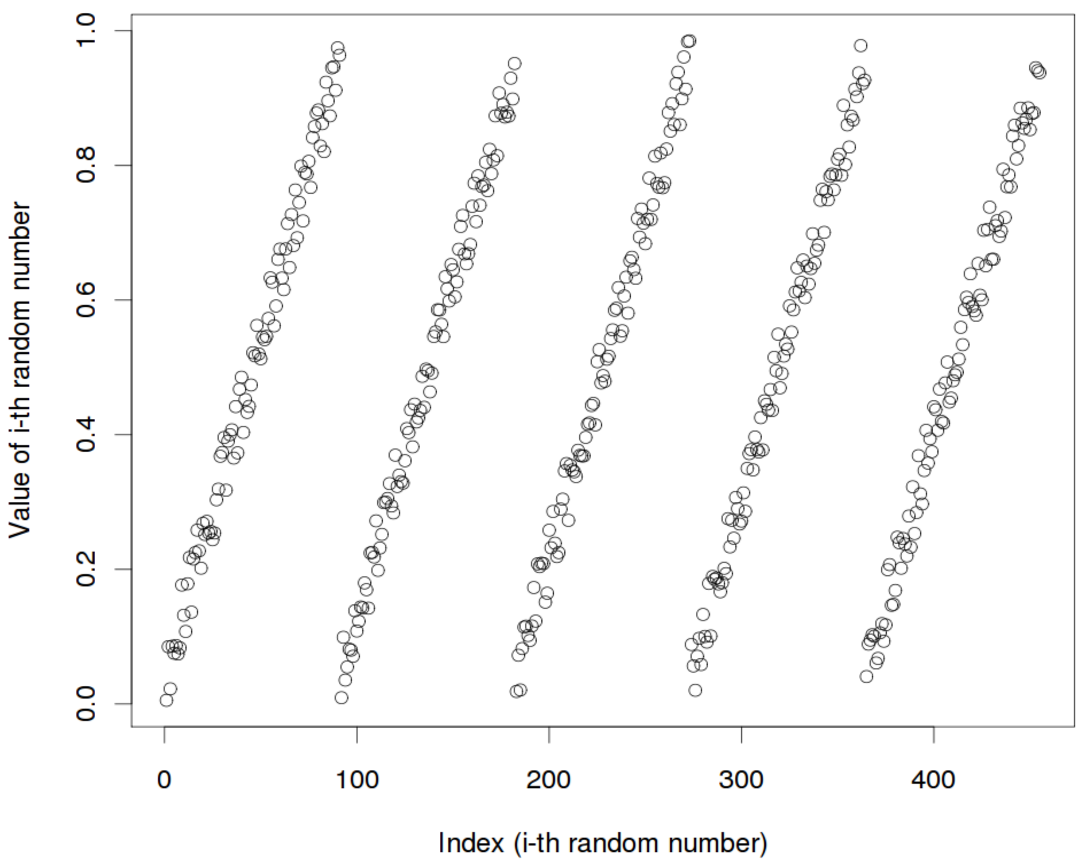
- How many intervals (k)?
  - A difficult problem for the general case
  - Warning: A smaller or a greater k may change the outcome of the test!
  - As a general rule, use k>100
  - As a general rule, make the intervals equal-sized
  - As another general rule, make sure that $\forall j$: $np_j \geq 5$
    (i.e., have enough samples that we expect to have at least 5 samples in each interval)
- ⇒ As a general rule, you need a lot of measurements!
- What confidence level?
  - At most $\alpha$=0.10 (almost too much);
    typical values: 0.001, 0.01, 0.05 [ , and 0.10]
  - The smaller, the better confidence in the test result

# Alternatives to test

- Kolmogorov-Smirnov test (K-S test)
  - Another very popular test
  - Advantages:
    - No grouping into intervals required
    - Valid for any sample size, not only for large n
    - More powerful than $\chi^2$ for a number of distributions
  - Disadvantages:
    - Applicability more limited than $\chi^2$
    - Difficult to apply to discrete data
    - If distribution needs to be fitted (unknown parameters), then K-S works only for a number of distributions
- Anderson-Darling test (A-D test)
  - Higher power than K-S for some distributions
- …a lot of other tests

# Tests for uniformity: limitations

- Consider this sequence of drawn “random numbers”:




- They are in U(0,1) … but do they seem random!?

# Recall our requirements for RNG

- RNs have to be uncorrelated — how to test this?
- Statistical tests:
  Draw some random numbers and examine them
  - Runs-up test
  - Serial test

- Theoretical parameters and theoretical tests:
  - Length of period
  - Spectral test
  - Lattice test

# Runs-up test

- Run up := the length of
  a contiguous sequence
  of monotonically increasing $X_i$.

- Example sequence:

| | |
|---|---|
| 0.86 > | length: 1 |
| 0.11 < 0.23 > | length: 2 |
| 0.03 < 0.13 > | length: 2 |
| 0.06 < 0.55 < 0.64 < 0.87 > | length: 4 |
| 0.10 | length: 1 |

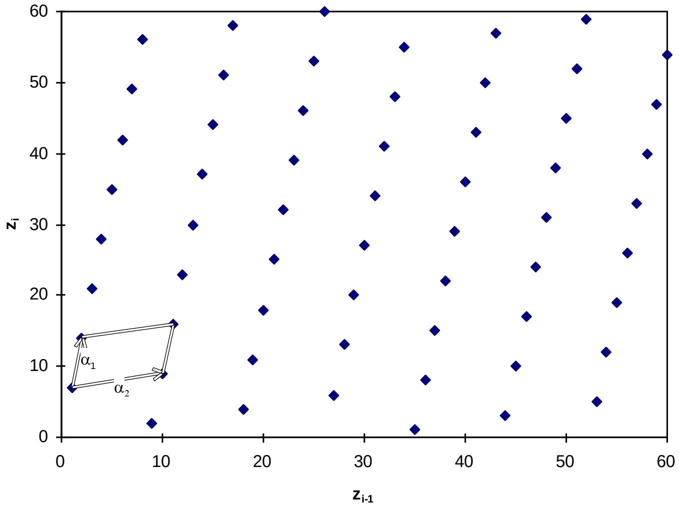
- Calculate $r_i$ (number of runs up of length i)
- Compute a test statistic value R, using the $r_i$ and
  a bestranging zoo of esoteric constants $a_{ij}$ and $b_j$
- R will have an approximate $\chi^2$ distribution with 6 df.

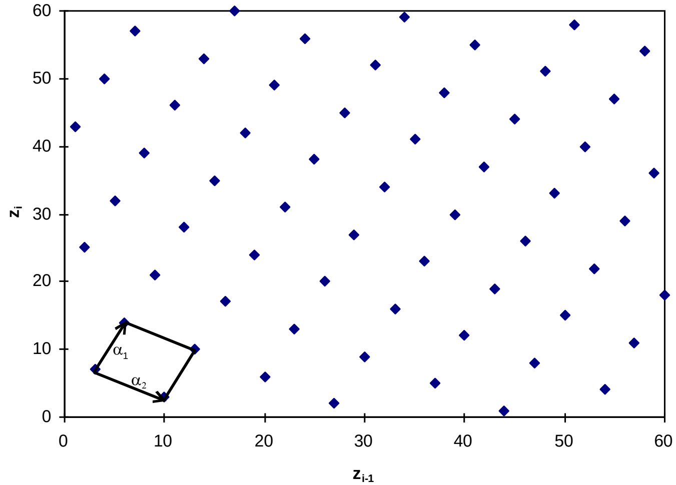
# Serial test

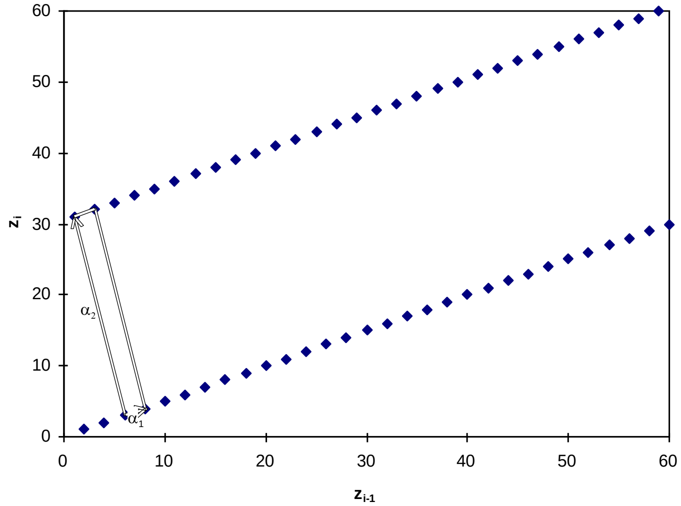
- Find possible correlations between subsequently drawn values
- Visual “tests”:
  - 2D plot of $X_i$ and $X_{i-1}$
  - 3D plot of $X_i$ and $X_{i-1}$ and $X_{i-2}$
- Generalisation: Serial test

# LCG examples (1/5)

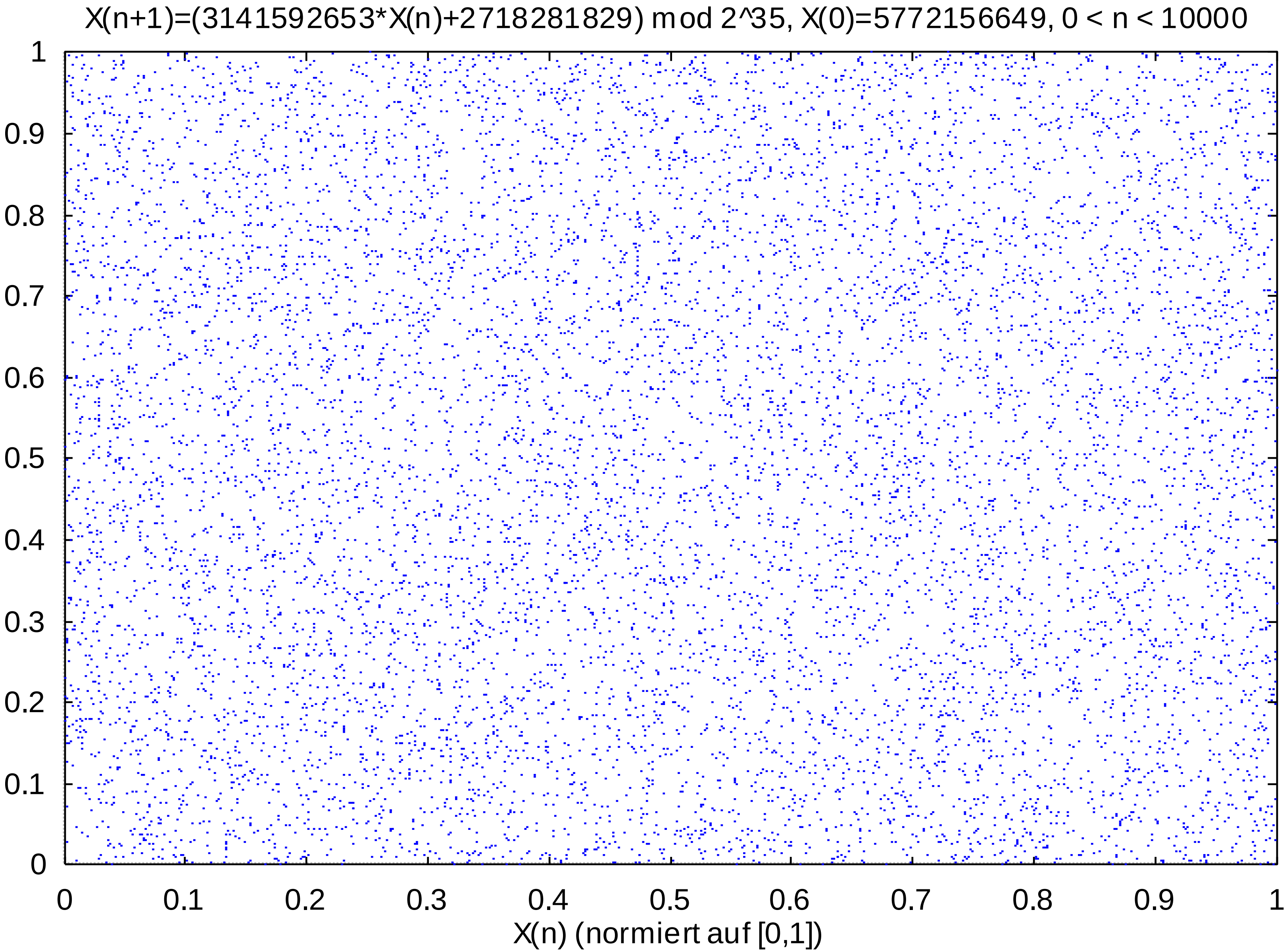
$$Z_i = a \cdot Z_{i-1} \quad (\mathrm{mod}\ 61)$$



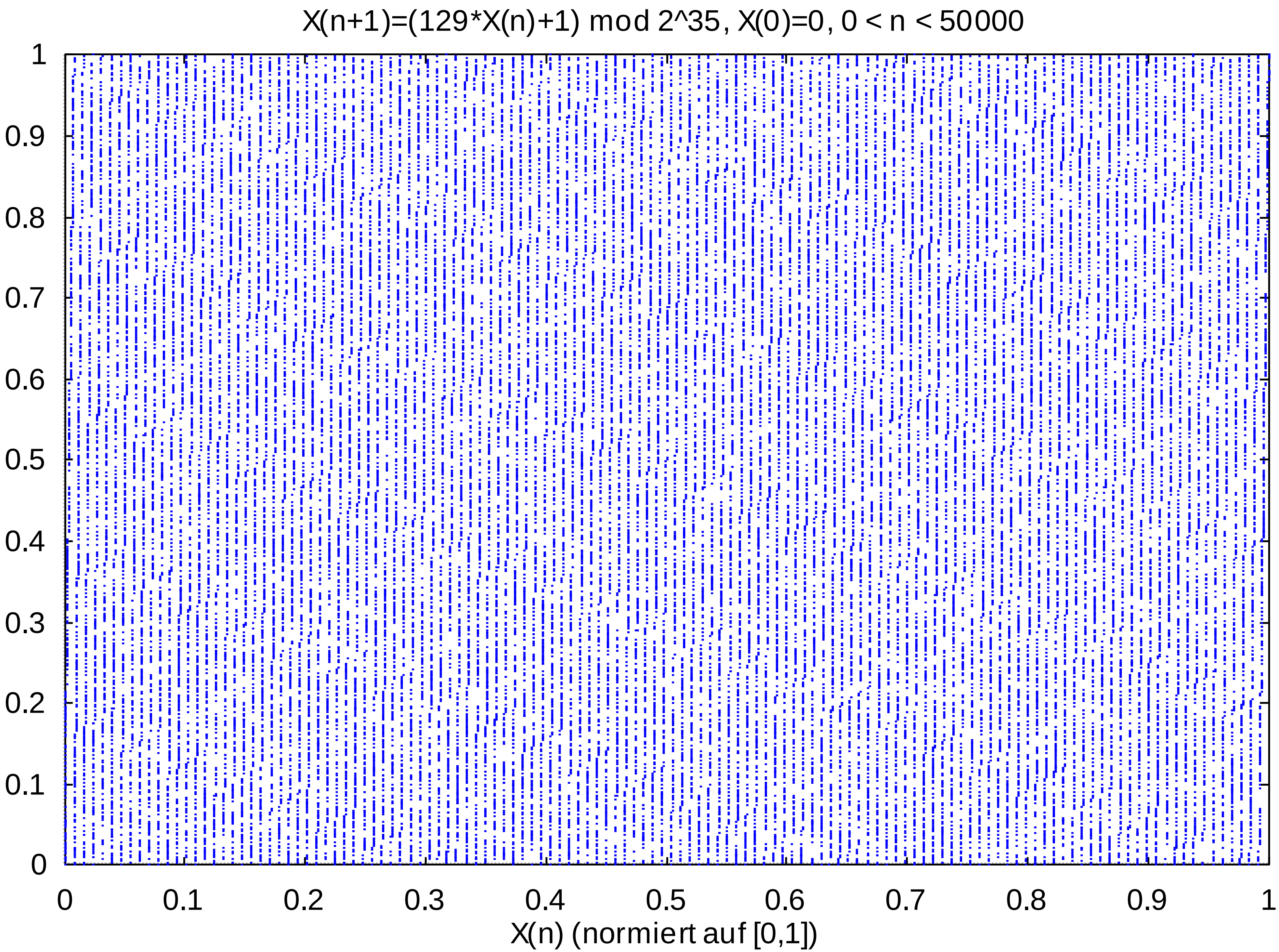



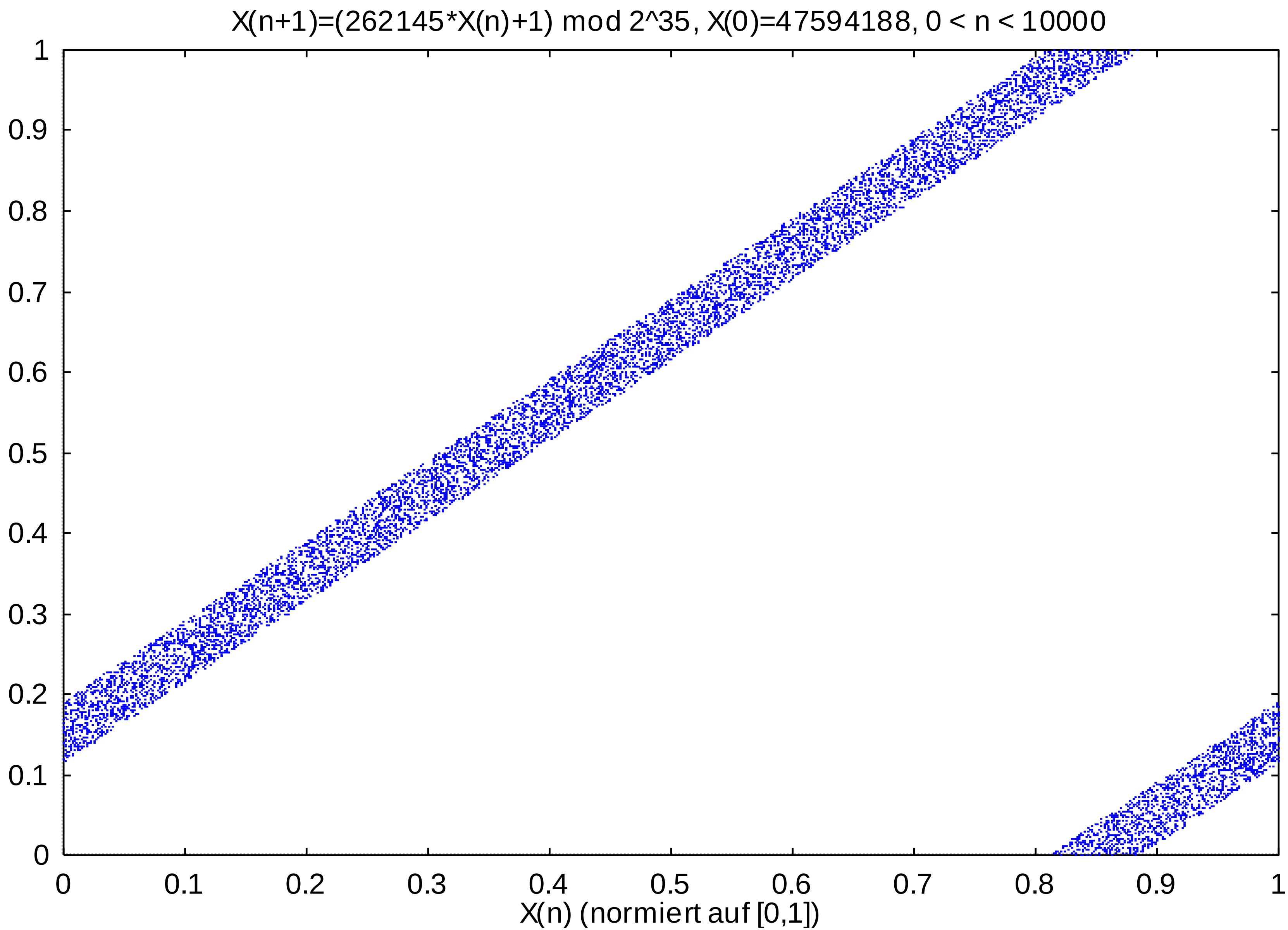
# LCG examples (2/5)



# LCG examples (3/5)

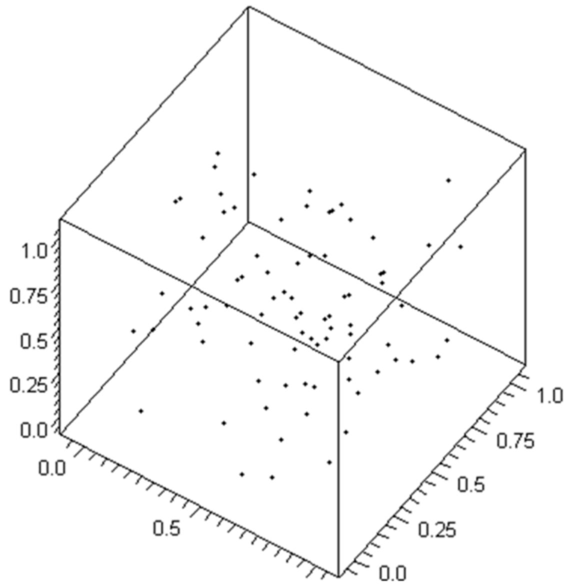


# LCG examples (4/5)

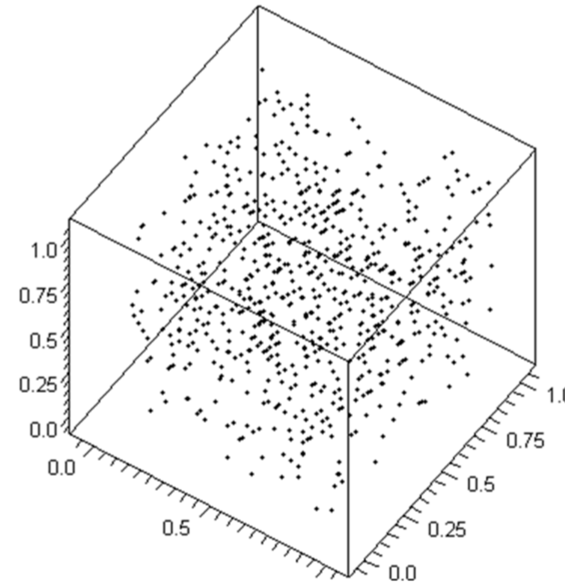


# LCG examples (5/5)

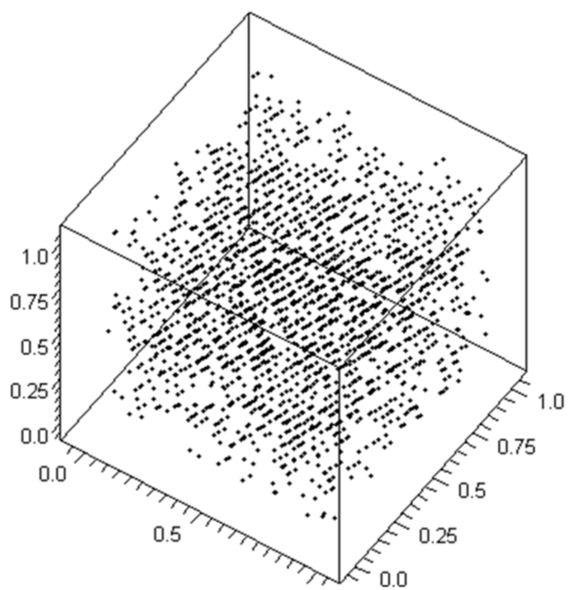
n=81



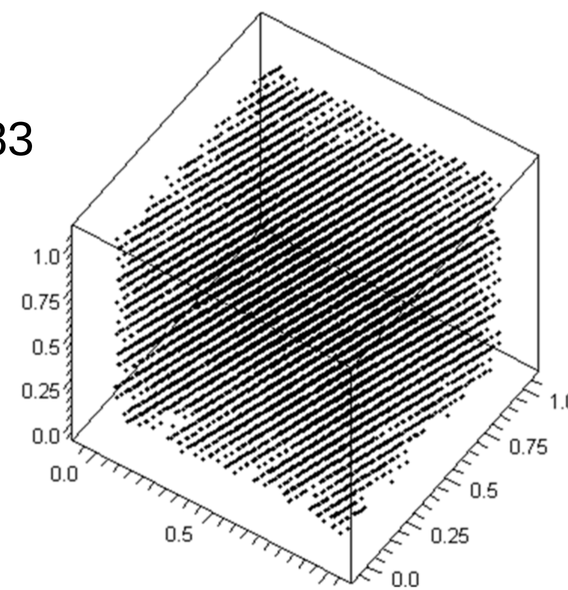

n=729




n=2197




n=19683



# Serial test

"a generalised and formalised version of the plots"

- Consider non-overlapping d-tuples of subsequently drawn random variables $X_i$ :
  $U_1 = (X_1, X_2, \ldots X_d)$  $U_2 = (X_{d+1}, X_{d+2}, \ldots, X_{2d})$  …
- These $U_i$'s are vectors in the d-dimensional space
- If the $X_i$ are truly iid random variables, then the $U_i$ are truly random iid vectors in the space $[0\ldots1]^d$
  (the d-dimensional hypercube)
- Test for d-dimensional uniformity (rough outline):
  - Divide [0…1] into k equal-sized intervals
  - Calculate a value $\chi^2(d)$ based on the number of $U_i$ for each possible interval combination
  - $\chi^2(d)$ has approximate distribution $\chi^2(k^d–1 \text{ df})$
  - Rest: same as $\chi^2$ test above

# RANDU

- A LCG with setup:
  $Z_i = 65{,}539 \cdot Z_{i-1} \bmod 2^{31}$
- Advantage: It's fast.
  - mod $2^{31}$ can be calculated with a simple AND operation
  - 65,539 is a bit more than $2^{16}$; thus the multiplication (=expensive operation) can be replaced by a bit shift of 16 bit plus three additions (=cheap operations)
  - Why 65,539? It's a prime number.
- Disadvantage:
  - An infamously bad RNG! Never, ever use it!
  - d≥3: The tuples are clumped into 15 plains (remember the animated 3D cube? That was RANDU!)
- A lot of simulations in the 1970s used RANDU
  ⇒ sceptical view on simulation results from that time

# Theoretical parameters, theoretical tests

- Tests so far: Based on drawing samples from RNG
- No absolute certainty!
  - Usually, only a small subset of entire period is used
  - Remember the $\chi^2$ test

- Theoretical parameters and tests
  - Based directly on the algorithm and its parameters
  - No samples to be drawn
  - Often complicated

# Period length

- After some time, the “random” numbers must repeat themselves. Why?
  - LCG: $Z_i$ is entirely determined by $Z_{i-1}$
  - The same $Z_{i-1}$ will always produce the same $Z_i$
  - There are only finitely many different $Z_i$
  - How many?
    We take mod m $\Rightarrow$ at most m different values
- Call this the period length

# Theorem by Hull and Dobell 1962

- A LCG has full period if and only if the following three conditions hold:
  1. c is relatively prime to m
     (i.e., they do not have a prime factor in common)
  2. If m has a prime factor q,
     then (a—1) must have a prime factor q, too
  3. If m is divisible by 4,
     then (a—1) must be divisible by 4, too
- ⇒Prime numbers play an important role
  - Remember RANDU?
    At least, it used a prime number…
- Multiplicative RNGs (i.e., no increment $Z_i$+c) cannot have period m.
  (But period (m—1) is possible if m and a are chosen carefully.)

# LCG and period length considerations

- On 32 bit machines, $m \leq 2^{31}$ or $m \leq 2^{32}$ due to efficiency reasons $\Rightarrow$ period length 4.3 billion
- Calculating that many random numbers only takes a couple of seconds on today's hardware
- Theory suggests to use only $\sqrt{period\_length}$ numbers; that's only 65,000 random numbers
- How many random numbers do we need? Example:
  - Simulate behaviour of 1,000 Web hosts
  - Each host consumes on average 1 random number per simulation second
  - Result: We can only simulate for one minute!
- We need much longer period lengths

# Spectral test (coarse description)

- ~ The theoretical variant of the serial test
- Observation by Marsaglia (1968):
  "Random numbers fall mainly in planes."
  - Subsequent overlapping (!) tuples $U_{i:}$
    $U_1=(X_1, X_2, \ldots X_d)$ $U_2=(X_2, X_3, \ldots, X_{d+1})$ …
    fall on a relatively small number of (d–1)-dimensional hyperplanes within the d-dimensional space
  - Note the difference to the serial test! (overlapping)
  - 'Lattice' structure
- Consider hyperplane families that cover all tuples $U_i$
- Calculate the maximum distance between hyperplanes. Call it $\delta_d$.
- If $\delta_d$ is small, then the generator can ~uniformly fill up the d-dimensional space

# Spectral test and LCG

- For LCG, it is possible to give a theoretical lower bound $\delta_d$*:
  $\delta_d \geq \delta_d^* = 1 / (\gamma_d\ m^{1/d})$
- $\gamma_d$ is a constant whose exact value is only known for d≤8 (dimensions up to 8)
- LCG do not perform very well in the spectral test:
  - All points lie on at most $m^{1/n}$ hyperplanes (Marsaglia's theorem)
  - Serial test: similar
  - There are way better random number generators than linear congruential generators.

# Discussion of LCGs

- Advantages:
  - Easy to implement
  - Reproducible
  - Simple and fast
- Disadvantages:
  - Period (length of a cycle) depends on parameters a, c, and m
  - Distribution and correlation properties of generated sequences are not obvious
  - A value can occur only once per period (unrealistic!)
  - By making a bad choice of parameters, you can screw up things massively
  - Bad performance in serial test / spectral test even for good choice of parameters

# Beyond LCGs

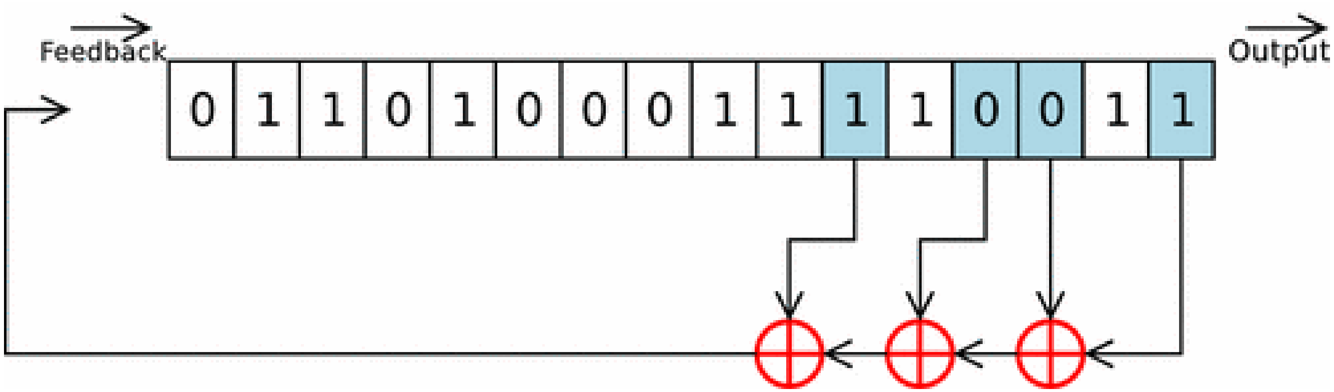
- Why linear?
  - Quadratic congruential generator:
    $$Z_i = (a \cdot (Z_{i-1})^2 + a' \cdot Z_{i-1}) \bmod m$$
  - Period is still at most m
- Why only use one previous $X_i$?
  - Multiple recursive generator:
    $$Z_i = (a_1 Z_{i-1} + a_2 Z_{i-2} + a_3 Z_{i-3} + \ldots + a_q Z_q) \bmod m$$
  - Period can be $m^q-1$ if parameters are chosen properly
- Why not change multiplier a and increment c dynamically, according to some other congruential formula?
  - Seems to work alright

# Feedback Shift Register Generators (1/2)

- Linear feedback shift register generator (LFSR) introduced by Tausworthe (1965)
- Operate on binary numbers (bits), not on integers
- Mathematically, a multiple recursive generator:

$$b_i = (c_1 b_{i-1} + c_2 b_{i-2} + c_3 b_{i-3} + \ldots + c_q b_q) \bmod 2$$

  - $c_i$: constants that are either 0 or 1
  - $c_q = 1$ (why?)
  - Observe that + mod 2 is the same as XOR (makes things faster)
- In hardware:



# Feedback Shift Register Generators (2/2)

- Usually only two cj coefficients are 1, thus:

$$b_i = (b_{i-r} + b_{i-q}) \bmod 2$$

- LFSR create random bits, not integers
  - Concatenate ℓ bits to form an ℓ-bit integer:

    $W_i = b_{(i-1)\ell+1}\, b_{(i-1)\ell+2} \ldots b_{i\ell}$

- Properties
  - Period length of the bi = $2^q-1$ if parameters chosen accordingly (Note: characteristic polynomial has to be primitive over Galois field $\mathcal{F}_2$ …)
  - Period length of the generated ints accordingly lower?
    - Depends on whether $\ell \mid 2^q-1$ or not—probably not the case
    - But there may be some correlation after one period
  - Statistical properties not very good
  - Combining LFSRs improves statistics and period

# Generalised feedback shift register (GFSR)

- Lewis and Payne (1973)

- To obtain sequence of $\ell$-bit integers $Y_1$, $Y_2$, …:
  - Leftmost bit of $Y_i$ is filled with LFSR-generated bit $b_i$
  - Next bit of Yi is filled with LFSR-generated bit after some “delay” d: $b_{i+d}$
  - Repeat that with same delay for remaining bits up to length $\ell$

- Mathematical properties
  - Period length can be very large if q is very large, e.g., Fushimi (1990): period length = $2^{521}–1 = 6.86 \cdot 10^{156}$
  - If $\ell$<q, then many $Y_i$’s will repeat during one period run (Is that good or bad?)
  - If two bits (as with LFSR), then $Y_i = Y_{i–r} \oplus Y_{i–q}$

# Mersenne Twister (1/2)

- Before we go into the mathematical details…
  - Very, very long period length: $2^{19,937}–1 > 10^{6,000}$
  - Very good statistical properties: OK in 623 dimensions
  - Quite fast
- State of the art: One of the best we have right now
  - The RNG of choice for simulations
  - Default RNG in Python, Ruby, Matlab, GNU R
  - Admittedly, there are even (slightly) better RNGs, cf. TestU01 paper
- Two warnings:
  - Not suitable for cryptographic applications:
    Draw 624 random numbers and you can predict all others!
  - Can take some time (“warm-up period”) until the stream generates good random numbers
    - Usually hidden from programmer through library
    - If in doubt, discard the first 10,000 drawn numbers

# Mersenne Twister (2/2)

- Twisted GFSR (TGFSR)
  - Matsumoto, Kurita (1992, 1994)
  - Replace the recurrence of the GFSR by

    $$Y_i = Y_{i-r} \oplus A \cdot Y_{i-q}$$

    where:
    - the $Y_i$ are $\ell$ x 1 binary vectors
    - A is an $\ell$ x $\ell$ binary matrix
  - Period length = $2^{q\ell}-1$ with suitable choices for r, q, A
- Mersenne Twister (MT19937)
  - Matsumoto, Nishimura (1997, 1998)
  - Clever choice of r, q, A and the first $Y_i$ to obtain good statistical properties
  - Period length $2^{19,937}-1 = 4.3 \cdot 10^{6001}$ (Mersenne prime: $2^n-1$)

# Digression: Period lengths revisited

What period lengths do we actually require?

- Estimate #1:
  - A cluster of 1 million hosts
  - each of which draws 1,000,000 · $2^{32}$ per second (~1,000,000 times as fast as today's desktop PCs)
  - for ten years

  will require…
  - 5.6 · $10^{27}$ random numbers
  - (Make the PCs again $10^6$ times faster ⇒ 5.6 · $10^{33}$)
- Estimate #2: What's the estimated number of electrons within the observable universe (a sphere with a radius of ~46.5 billion light years)
  - About $10^{80}$ (± take or leave a few powers of 10)

# Test batteries

- A lot of tests, a lot of different RNGs
- How to compare them?
- Benchmark suites (‘Test batteries’)
  that bundle many statistical tests:
  - TestU01 (L’Ecuyer)
  - DIEHARD suite (Marsaglia)
  - NIST test suite (National Institute of Standards and Technologies; ≙ Physikalisch-Technische Bundesanstalt)

# Conclusion: Quality tests for RNG

- Empirical tests (based on generated samples)
  - For U(0,1) distribution: $\chi^2$ test
  - For independence: autocorrelation, serial, run-up tests
- Theoretical tests (based on generation formula)
  - Basic idea: test for k-dimensional uniformity
  - Points of sequence form system of hyperplanes
  - Computation of distance of hyperplanes for several dimensions k
  - Rather difficult optimization problem
- Conclusion
  - Implement/use only tested random number generators from literature, no "home-brewed" generators!
  - When in doubt, use the Mersenne Twister
    (but not for cryptography!)

# RNG: outlook

- A wide research field, still somewhat active
  - Many more algorithms exist
  - Many more tests for randomness exist
  - More are being developed
- If you are interested in this topic, you might want to have a look at this quite readable paper:
  - L'Ecuyer, Simard
    TestU01: a C library for empirical testing of random number generators
    ACM Transactions on Mathematical Software,
    Volume 33, No. 4, 2007