Discrete Event Simulation

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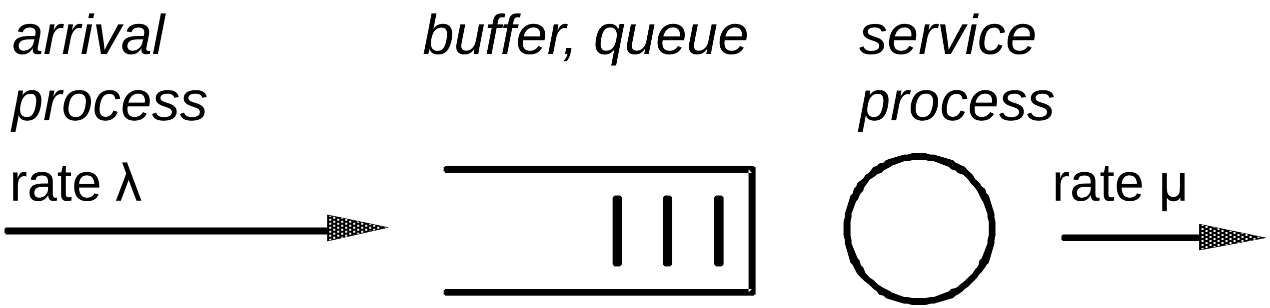
Topics

- Waiting Queues
- Random Variable
- Probability Space
- Discrete and Continuous RV
- Frequency Probability(Relative Häufigkeit)
- Distribution(discrete)
- Distribution Function(discrete)
- PDF,CDF
- Expectation/Mean, Mode,
- Standard Deviation, Variance, Coefficient of Variation
- p-percentile(quantile), Skewness, Scalability Issues(Addition)
- Covariance, Correlation, Autocorrelation
- Visualization of Correlation
- PP-Plot
- QQ-Plot

Waiting Queue Theory



3.4
0.6
1.7
1.6
0.7
0.7
1.3

inter-arrival times

random numbers

1.7
1.0
3.3
4.0
0.7
...

service times

Statistics Fundamentals

What are we talking about… and why?

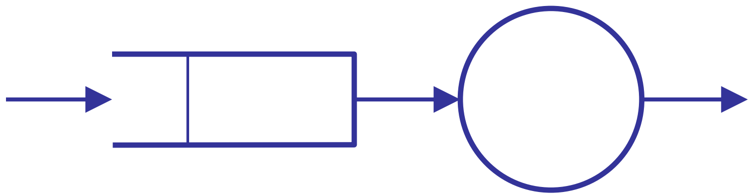
- Simple queue model:
  - Customers arrive at random times
  - Execution unit serves customers (random duration)
  - Only one customer at a time; others need to queue

- Standard example

- Give deeper understanding of important aspects, e.g.
  - Random distributions (input)
  - Measurements, time series (output)
  - …

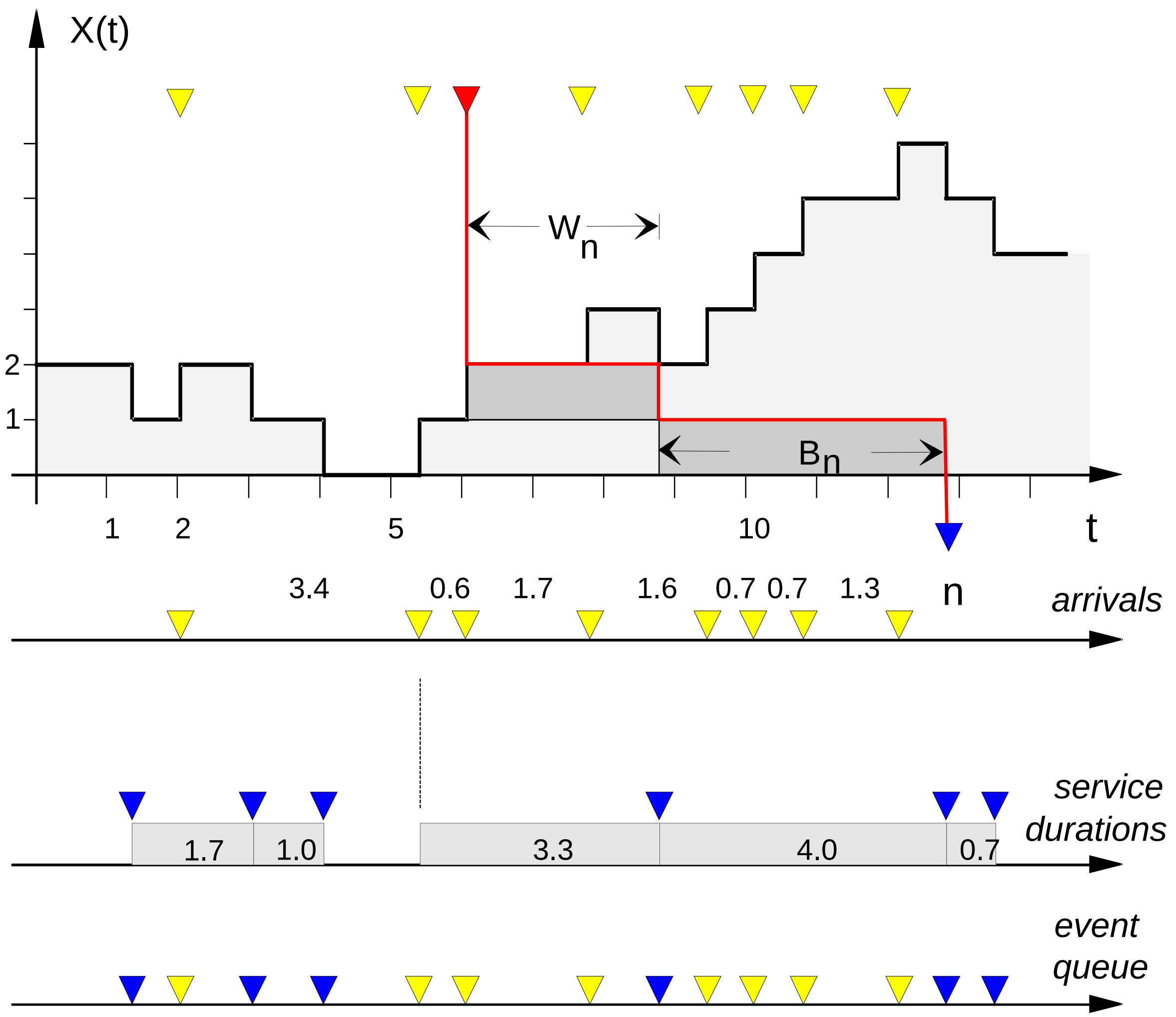
Queuing model: Input and output

- Input:
  - (Inter-)arrival times of customers (usually random)
  - Job durations (usually random)



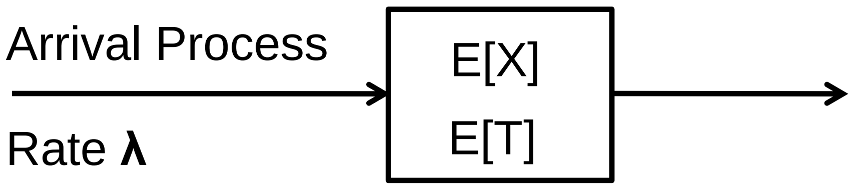
- Direct output:
  - Departure times of customers
- Indirect output:
  - Inter-arrival times for departure times of customers
  - Queue length
  - Waiting time in the queue
  - Load of service unit (how often idle, how often working)

Little Theorem



Little Theorem

- λ : average arrival rate
- E[X] : average number of packets in the system
- E[T] : average retention time of packets in the system



$$\boxed{t_o \to \infty}$$

$$\bar{T} = \frac{1}{N}\sum_{i=1}^{N} T_i \approx \frac{1}{N}\int_0^{t_o} X(t)dt \qquad \lambda = \lim_{t_o \to \infty} \bar{\lambda} = \lim_{t_o \to \infty} \frac{N}{t_o}$$

$$\bar{X} = \frac{1}{t_o}\int_0^{t_o} X(t)dt \Longrightarrow \bar{X} \approx \frac{N}{t_o} E[T] \qquad E[T] = \lim_{t_o \to \infty} \bar{T} = \lim_{t_o \to \infty} \frac{1}{N}\sum_{i=1}^{N} T_i$$

$$\bar{\lambda} \approx \frac{N}{t_o} \Longrightarrow \bar{\lambda} \cdot \bar{T} \approx \bar{X} \qquad E[X] = \lim_{t_o \to \infty} \bar{X} = \lim_{t_o \to \infty} \frac{1}{t_o}\int_0^{t_o} X(t)dt$$

- **Kendall Notation**

$$GI^{[x]} / GI / n - S$$

Number of Places in the Queue

$S = 0$ Loss/Blocking System

$S = \infty$ Waiting System

Number of Servers

Service Time Distribution

Batch Arrival Process

Arrival Process

Statistics Fundamentals

Queuing Discipline

FIFO / FCFS	First In First Out / First Come First Served
LIFO / LCFS	Last In First Out / Last Come First Served
SIRO	Service In Random
PNPN	Priority-based Service
EDF	Earliest Deadline First

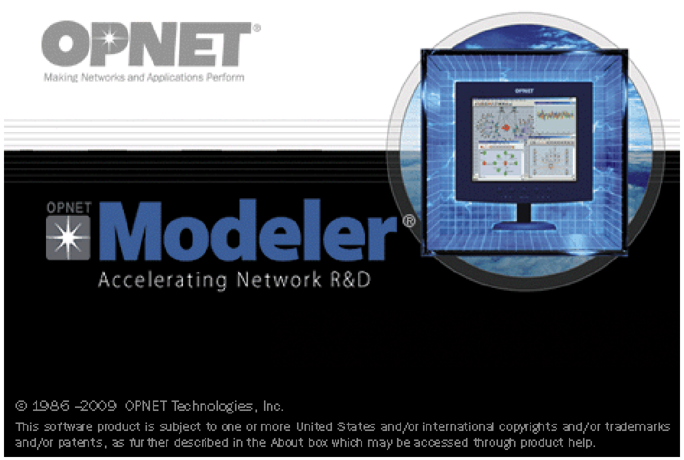
Distributions

M	Markovian	Exponential Service Time
D	Degenerate Distribution	A deterministic service time
E_k	Erlang Distribution	Erlang k distribution
GI	General distribution	General independent
H_k	Hyper exponential	Hyper k distribution

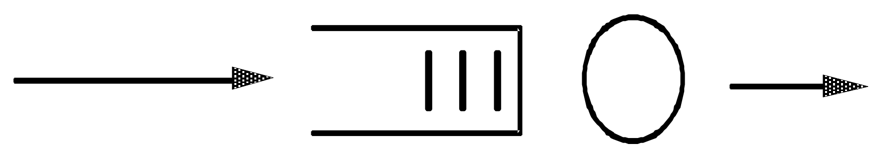
Statistics Fundamentals

- **System Characteristics**
  - Average customer waiting time
  - Average processing time of a customer
  - Average retention time of a customer
  - Average number of customers in the queue
  - Customer blocking probability
  - Utilization of the system / individual processing units

- Example



How to model and evaluate waiting queues in OPNET



Statistics Fundamentals

Exercise

- System A: D / D / 1 - ∞
  - Arrival rate $\lambda = 1$ / s
  - Service rate $\mu = [1;10]$ / s
- System B: M / M / 1 - ∞
  - Arrival rate $\lambda = 1$ / s
  - Service rate $\mu = [1;10]$ / s

- What is the maximum (meaningful) utilization of the system?
- Which system performs better?
- What impact does the utilization have on the system?

- System C: M / M / 20 - ∞
  - Arrival rate $\lambda = 10$ / s
  - Service rate $\mu = 1$ / s
- System D: M / M / 1 - ∞
  - Arrival rate $\lambda = 10$ / s
  - Service rate $\mu = 20$ / s

- Which system performs better?
- Would you prefer a single fast processing unit instead of multiple slow processing units?

Statistics Fundamentals

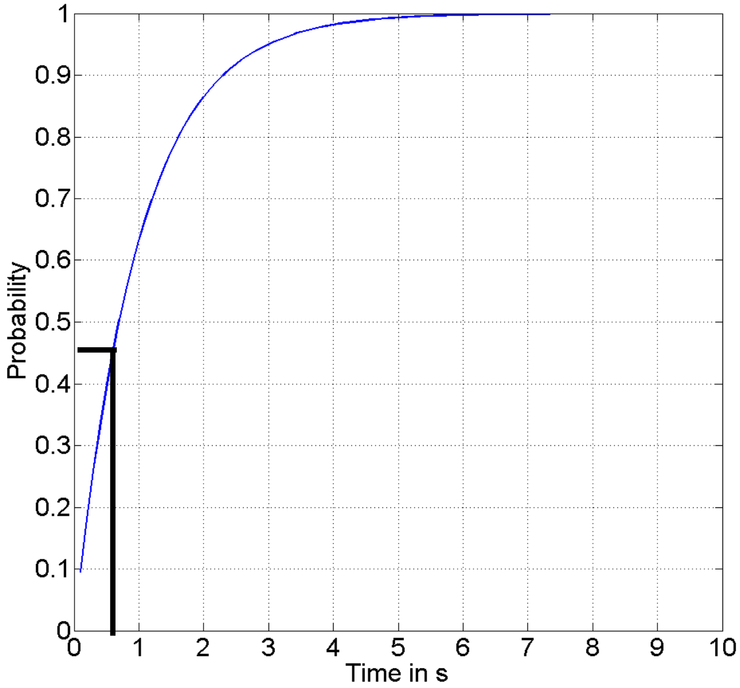
- Exercise
  - System E: M / M / 10 - ∞
    - Arrival rate $\lambda = 9$ / s
    - Service rate $\mu = 1$ / s
  - System F: M / M / 100 - ∞
    - Arrival rate $\lambda = 90$ / s
    - Service rate $\mu = 1$ / s

- What is the maximum (meaningful) utilization of the system?
- Which system performs better?

  - System G: M / D / 1 - ∞
    - Arrival rate $\lambda = 1$ / s
    - Service rate $\mu = 1 / 0.7$ / s
  - System H: D / M / 1 - ∞
    - Arrival rate $\lambda = 1$ / s
    - Service rate $\mu = 1 / 0.7$ / s

- Which system performs better?
- Which system has a shorter avg waiting time?

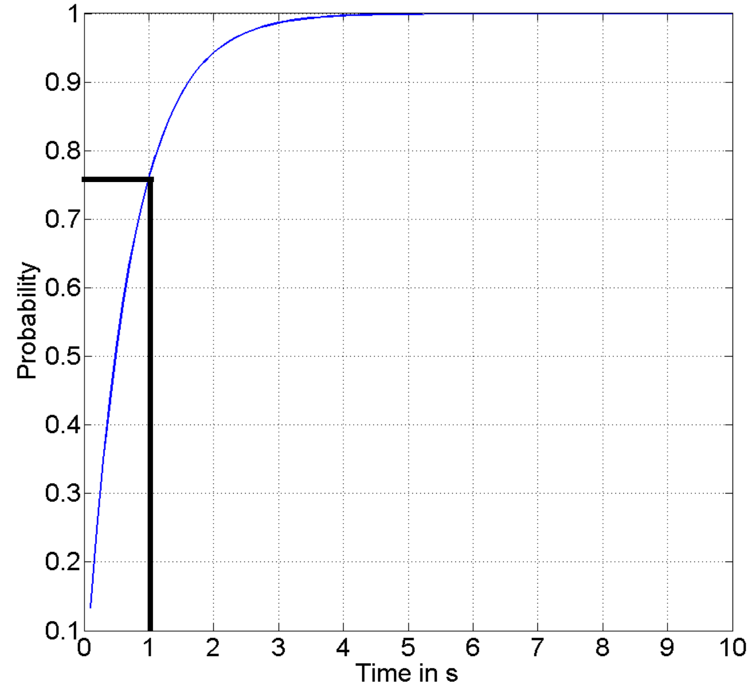
- System G: M / D / 1 - ∞



λ = 1

$$P(T > 0.7s) = 1 - P(T \leq 0.7s) = 0.55$$

- System H: D / M / 1 - ∞

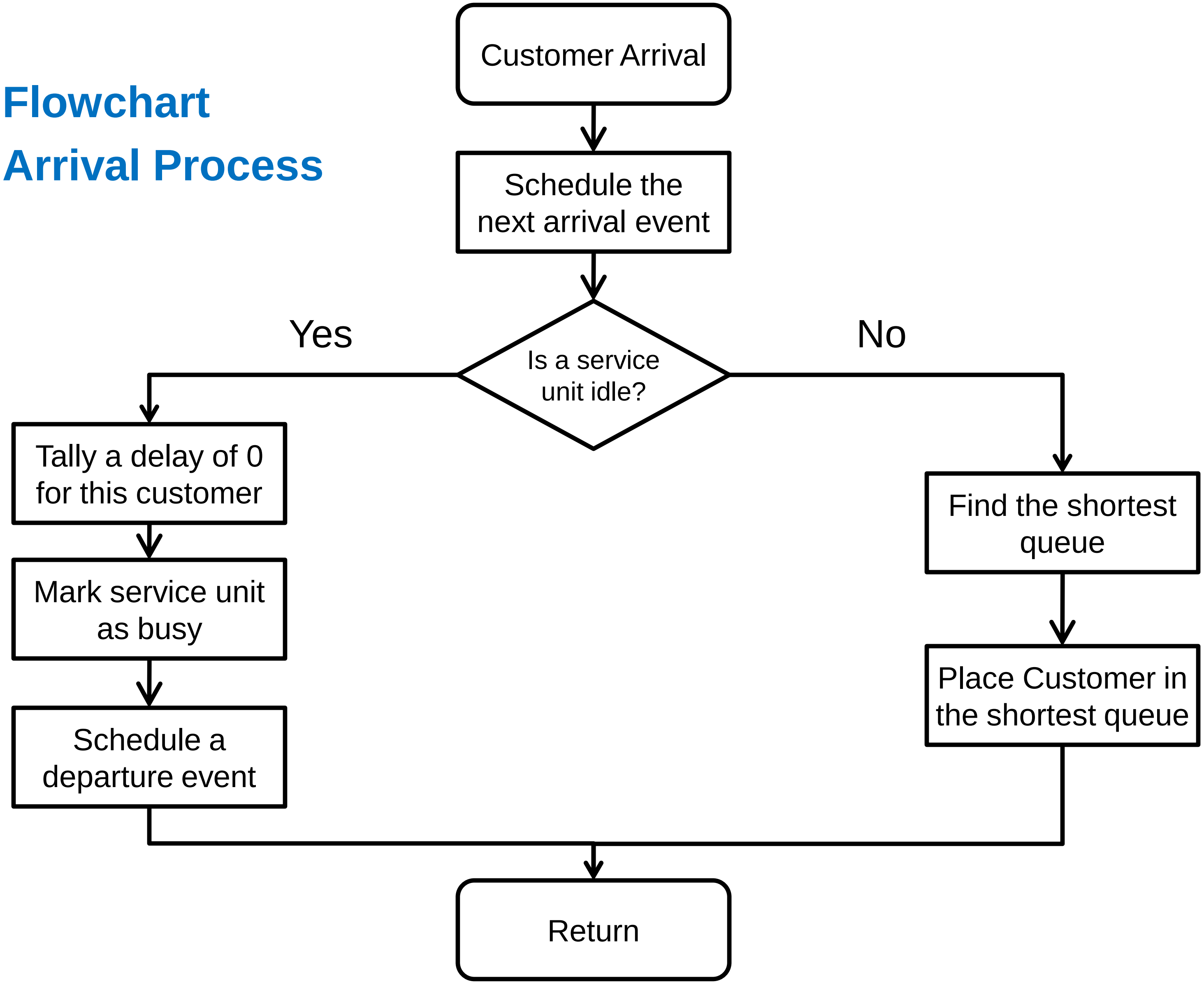


μ = 1 / 0.7 = ~ 1.43

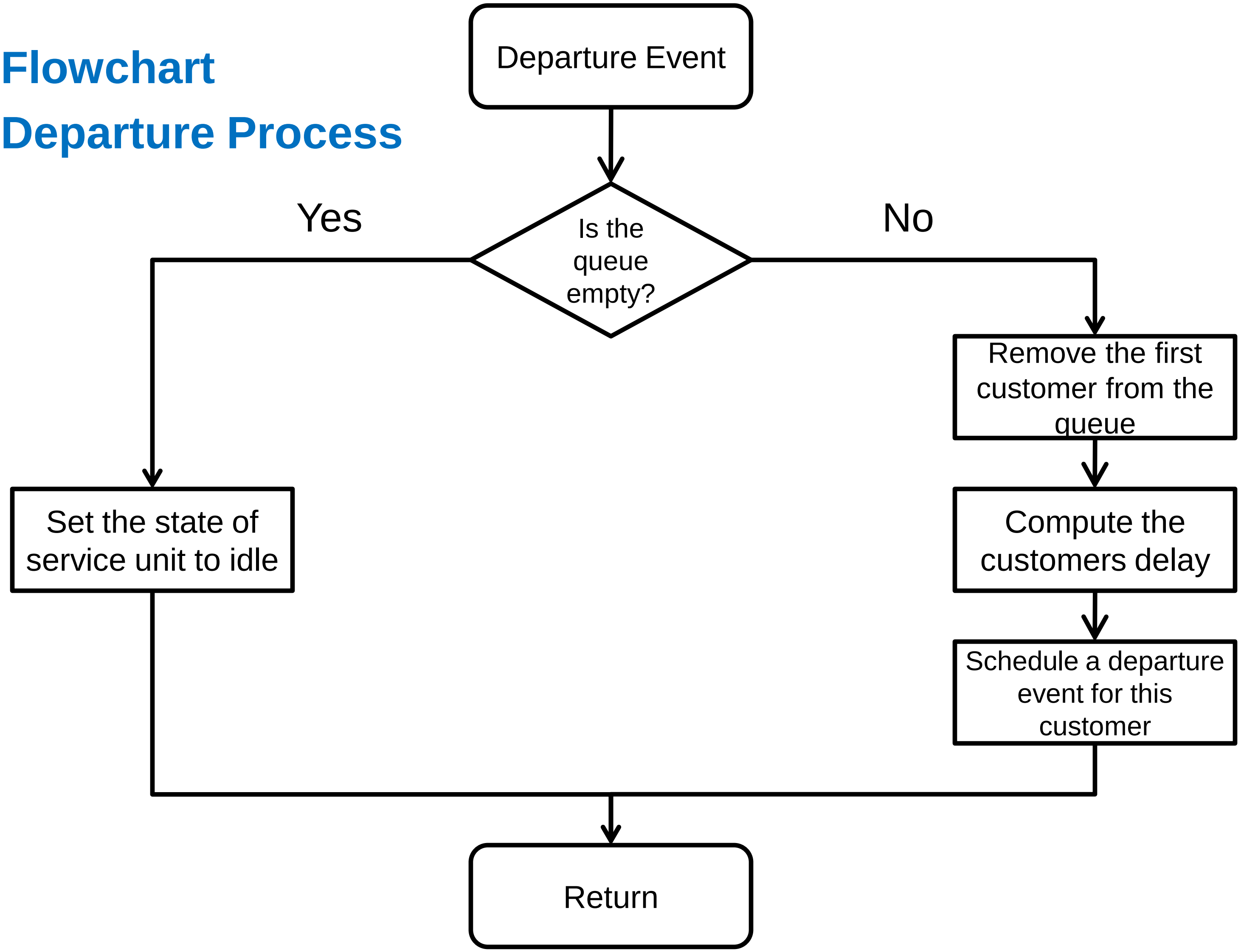
$$P(T < 1s) = 0.75$$

Flowchart

Arrival Process



Flowchart
Departure Process



Statistics Fundamentals

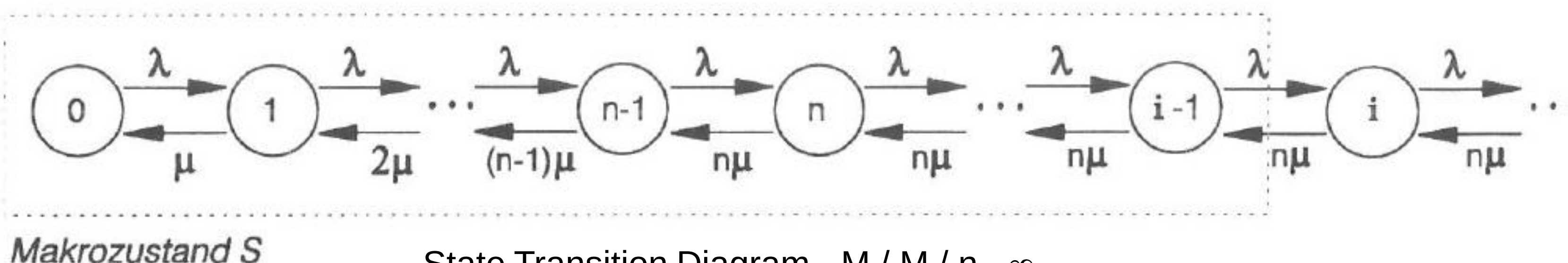
❑ M / M / n - ∞

- System of equations

$$\lambda x(i-1) = i\mu x(i), \qquad i = 1,2,3,\ldots,n,$$

$$\lambda x(i-1) = n\mu x(i), \qquad i = n+1,\ldots$$

$$\sum_{i=0}^{\infty} x(i) = 1$$

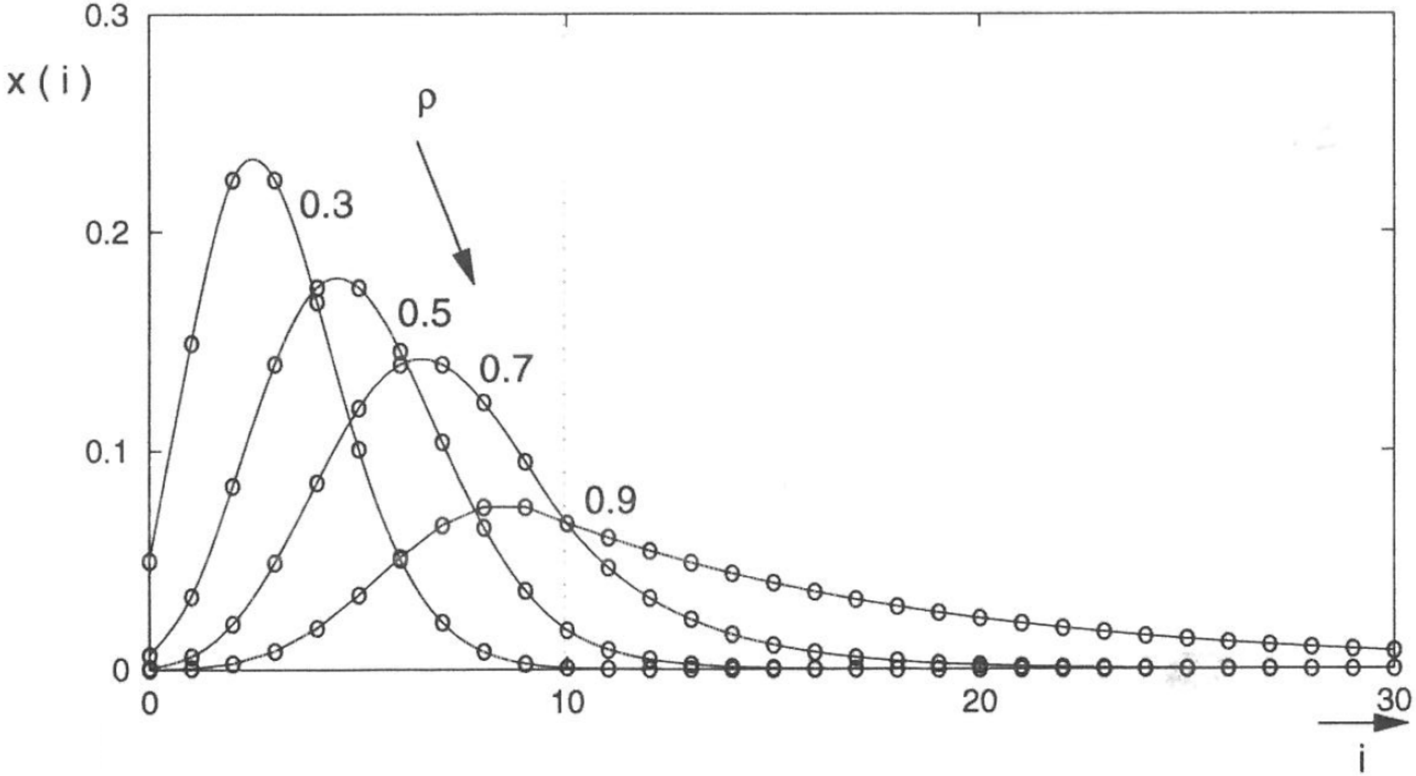


State Transition Diagram - M / M / n - ∞

Picture taken from Tran-Gia, Analytische Leistungsbewertung verteilter Systeme, p. 98

Statistics Fundamentals

- **M / M / 10 - ∞**

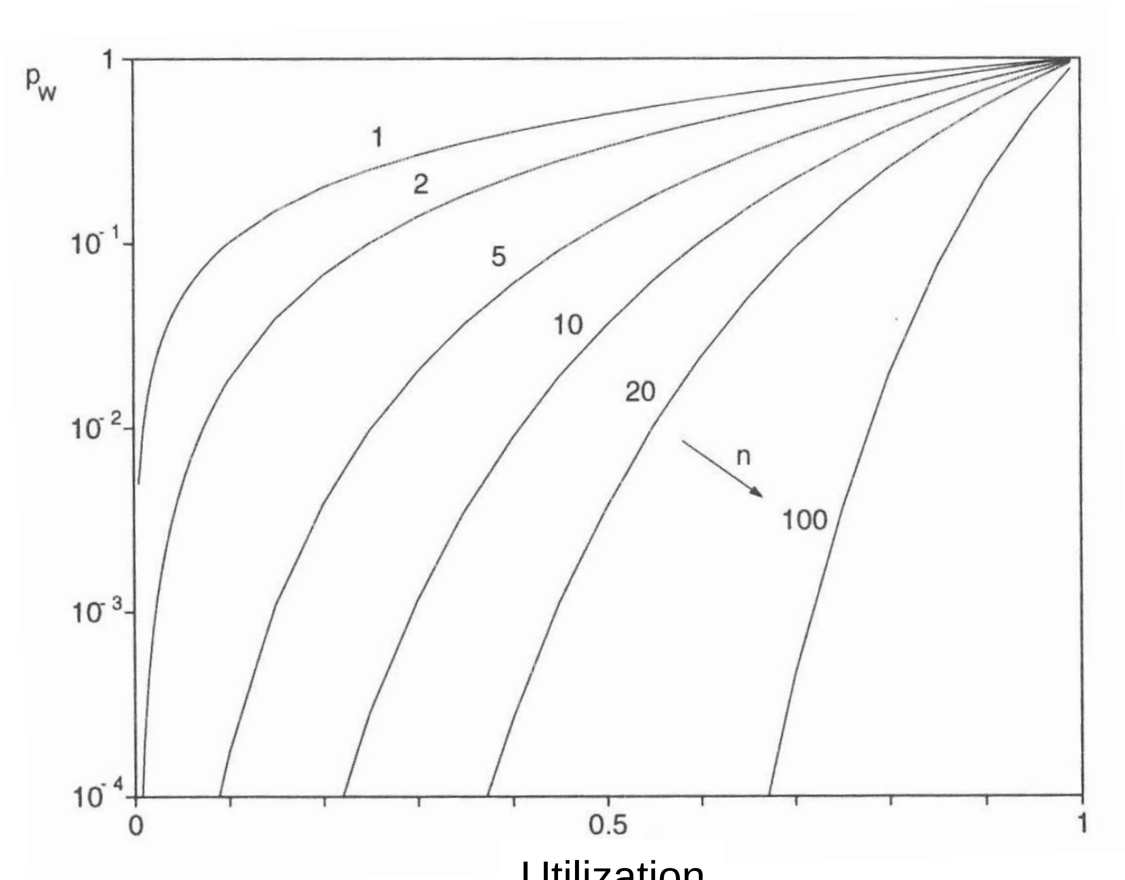


State Distribution - M / M / 10 - ∞

Picture taken from Tran-Gia, Analytische Leistungsbewertung verteilter Systeme, p. 99

Statistics Fundamentals

- **M / M / 10 - ∞**
  - **The waiting probability decreases with an increasing number of processing units (assuming constant utilization)**



Utilization

- n = number of processing units
- ρ_W - Waiting probability

Picture taken from Tran-Gia, Analytische Leistungsbewertung verteilter Systeme, p. 100

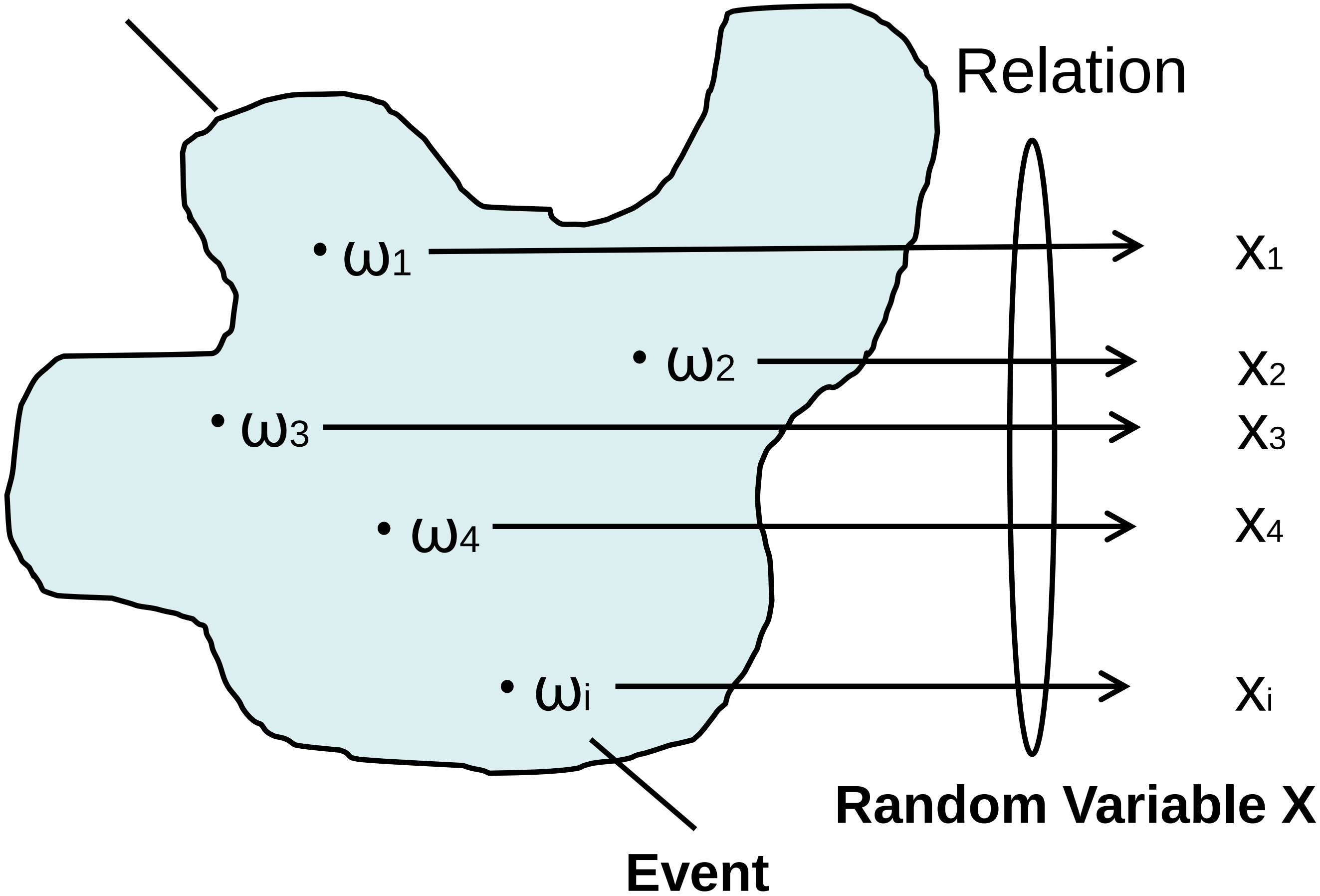
Classic definition of probability

The theory of chance consists in reducing all the events of the same kind to a certain number of cases equally possible, that is to say, to such as we may be equally undecided about in regard to their existence, and in determining the number of cases favorable to the event whose probability is sought. The ratio of this number to that of all the cases possible is the measure of this probability, which is thus simply a fraction whose numerator is the number of favorable cases and whose denominator is the number of all the cases possible.

- Pierre-Simon Laplace, A Philosophical Essay on Probabilities

- **Random Variable**

Probability Space **(Ereignisraum)** $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \dots, \omega_i\}$

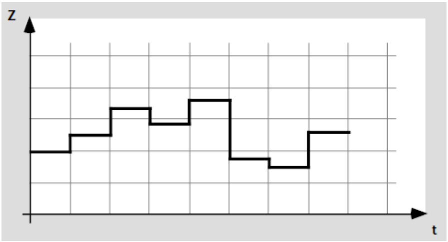


Statistics Fundamentals

- **Discrete Random Variable:**

Countable

  - Example: Flipping of a coin
    - ω_1={head-0}, ω_2={tail-1} => $\Omega = \{\omega_1, \omega_2\}$
    - X e {0, 1}

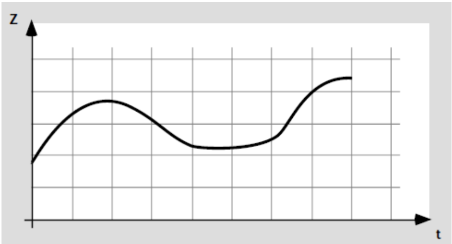


  - Example: Rolling two dice
    - ω_1={2}, ω_2={3}, … , ω_{11}={12} => $\Omega = \{\omega_1, \omega_2, \dots, \omega_{11}\}$
    - X e {2, 3, 4, …, 12}

- **Continuous Random Variable:**

Uncountable

  - Example: Round Trip Time
    - T e {5ms, 200ms}
    - ω_1={t<10ms}, ω_2={10ms≤t<20ms}, ω_3={t≥20ms} => $\Omega = \{\omega_1, \omega_2\}$



  - Example: Sensed Interference Level

Discrete or not discrete

Statistics Fundamentals

- **Frequency Probability** / Law of large numbers (Relative Häufigkeit)

  - Number of random experiments
    - n — total number of trials
    - X_i — event or characteristic of the outcome
    - n_i — number of trials where the event X_i occurred

$$h(X_i) = \frac{n_i}{n} \qquad 0 \le h(X_i) \le 1 \qquad \sum_i h(X_i) = 1$$

Vollständigkeits-relation

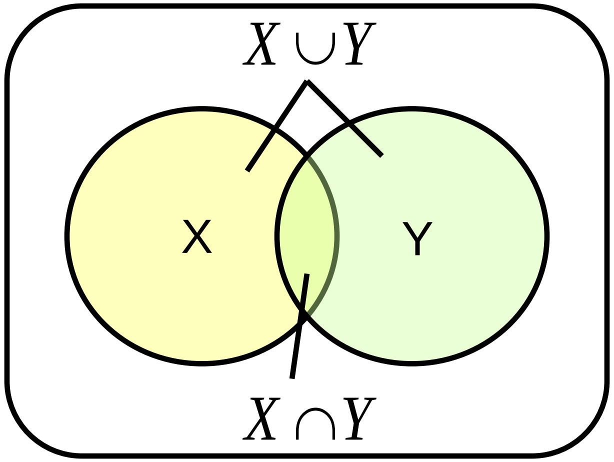
$$P(X_i) = \lim_{n \to \infty} \frac{n_i}{n} \qquad 0 \le P(X_i) \le 1 \qquad \sum_i P(X_i) = 1$$

X_i disjoint

Statistics Fundamentals

- **Vollständiges Ereignissystem**

$$P(Y) = \sum_{i=1}^{N} P(X_i)$$



- **Verbundereignis**

$$P(X \cap Y) = P(X, Y) = P(Y, X)$$

$$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$$

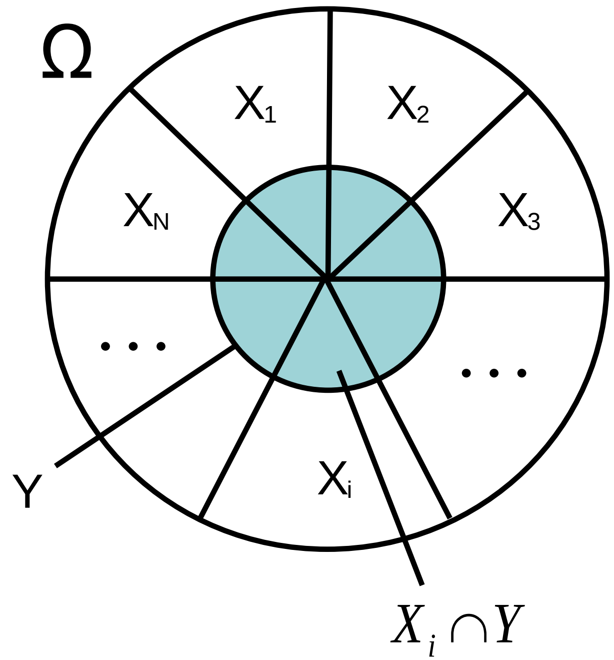
- **Bedingte Wahrscheinlichkeit**

$$P(X \mid Y) = \frac{P(X, Y)}{P(Y)} \quad \Rightarrow \quad P(X \mid Y) \geq P(X, Y)$$

- **Statistische Unabhängigkeit**

$$P(X \mid Y) = P(X) \qquad \text{v} \qquad P(X, Y) = P(X)P(Y)$$

- **Vollständiges Ereignissystem**
  - $$P(Y) = \sum_{i=1}^{N} P(X_i, Y)$$



- **Bayes Function**
  - $$P(X_i \mid Y) = \frac{P(Y \mid X_i) \cdot P(X_i)}{P(Y)} = \frac{P(Y \mid X_i) \cdot P(X_i)}{\sum_{k=1}^{N} P(Y \mid X_k) \cdot P(X_k)}$$

- **Distribution** (Verteilung)

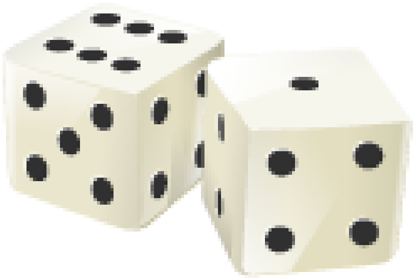
X – discrete random variable

- Function x(i) = P(X = i) , i = 0,1,2,…,X_{max} (Distribution)
  – x(i) e [0,1]
  – $\sum_{i=0}^{X_{max}} x(i) = 1$ (Vollständigkeitsrelation)

- Example:
  Rolling two dice
  • ω_1={2}, ω_2={3}, … , ω_{11}={12} => Ω = {ω_1, ω_2,… , ω_{11}}
  • X e {2, 3, 4, …, 12}

Statistics Fundamentals

- Example: Throwing two dice



Sample Space (Ereignisraum)

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

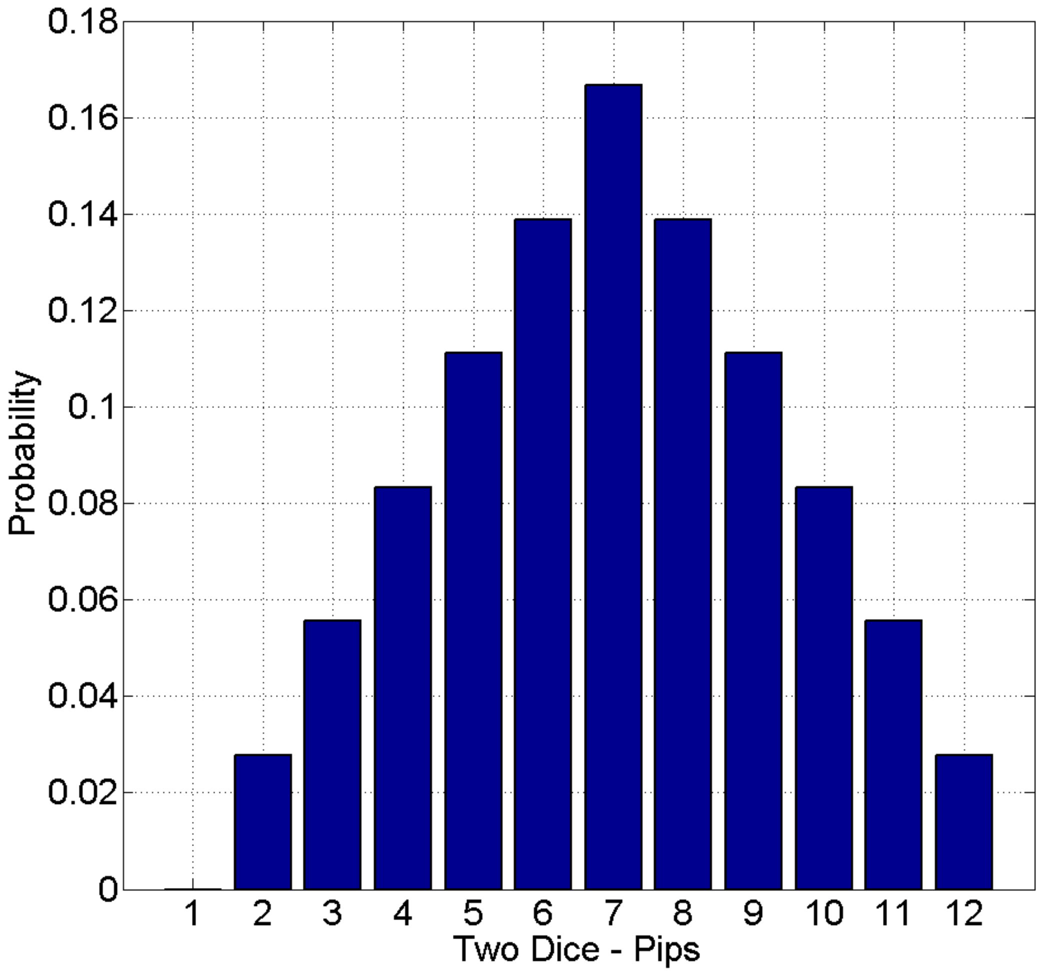
Statistics Fundamentals

- Example: Throwing two dice

Sample Space (Ereignisraum)

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Statistics Fundamentals

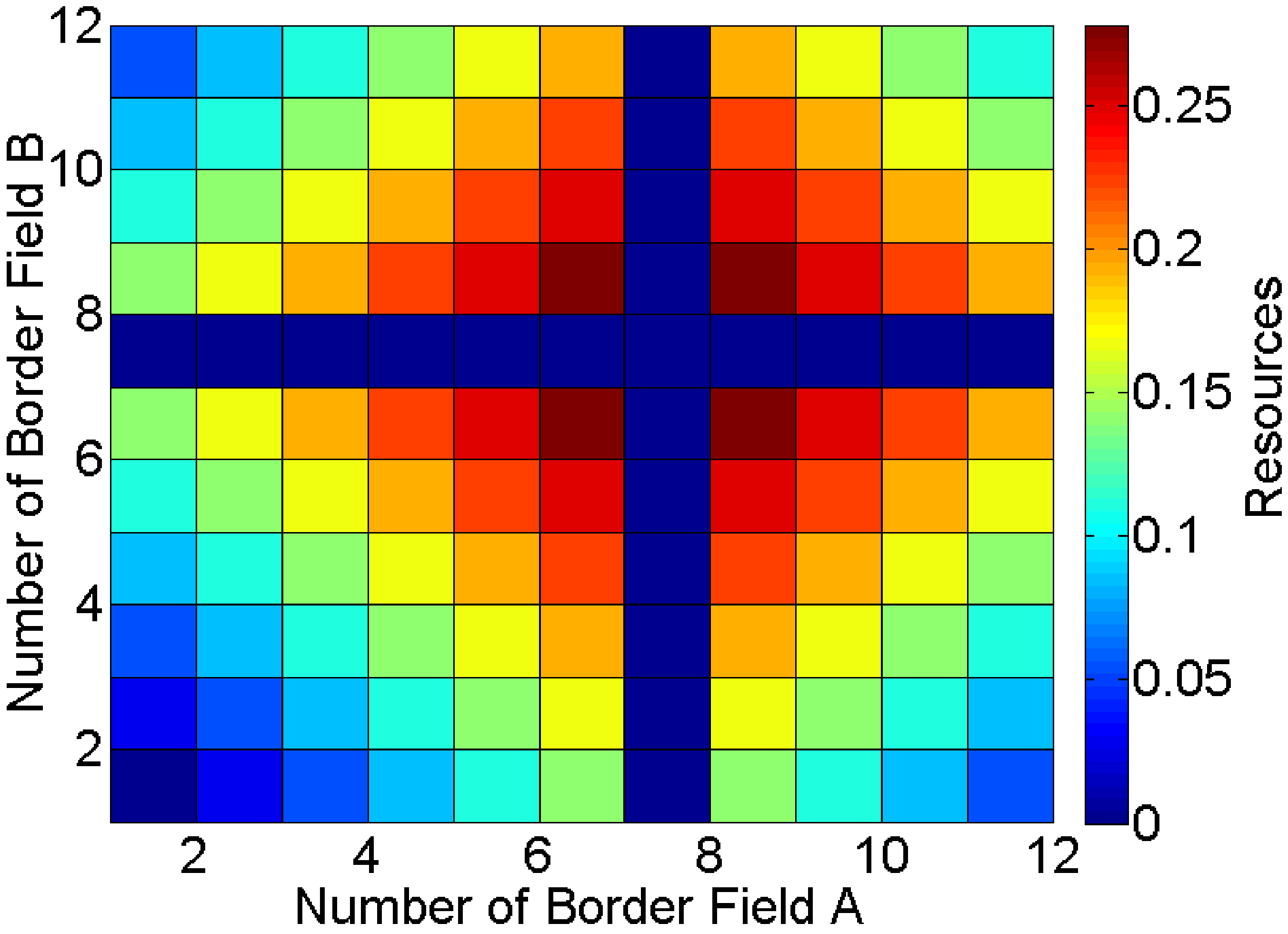


Distribution

Palour Game: Die Siedler von Catan

- Rules:
  - Players are only allowed to build along borders of a field
  - Players roll two dice
  - If the sum of the dice corresponds to the number of the field, the player gets the resources from this field
- Question
  - Where is the best place for a building?

Statistics Fundamentals



Statistics Fundamentals

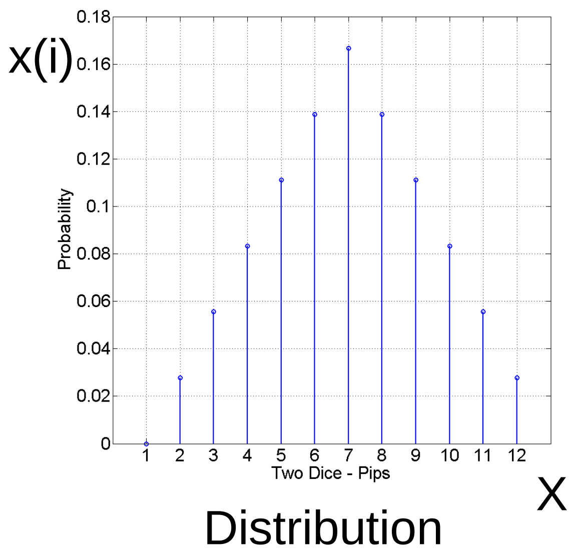
9
8
7
6
5
4
3
2
1

Prison

Are all fields reached with the same probability?

Statistics Fundamentals

- **Distribution** (Verteilung)
  - Discrete random variable X
  - i value of the random variable X
  - x(i) probability that the outcome of random variable X is i
  - $x(i) = P\{X = i\}, \quad i = 0,1,\ldots,X_{\max}$ (Distribution)
  - $\sum_{i=0}^{X_{\max}} x(1) = 1$ (Vollständigkeitsrelation)

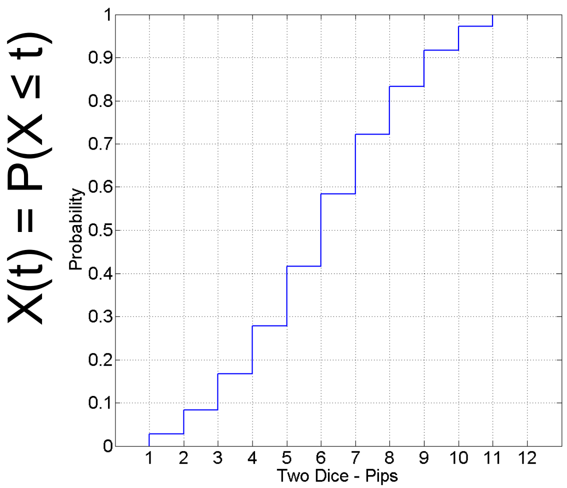


Statistics Fundamentals

- **Distribution function** (Verteilungsfunktion)

$$X(t) = P\{X \leq t\}$$

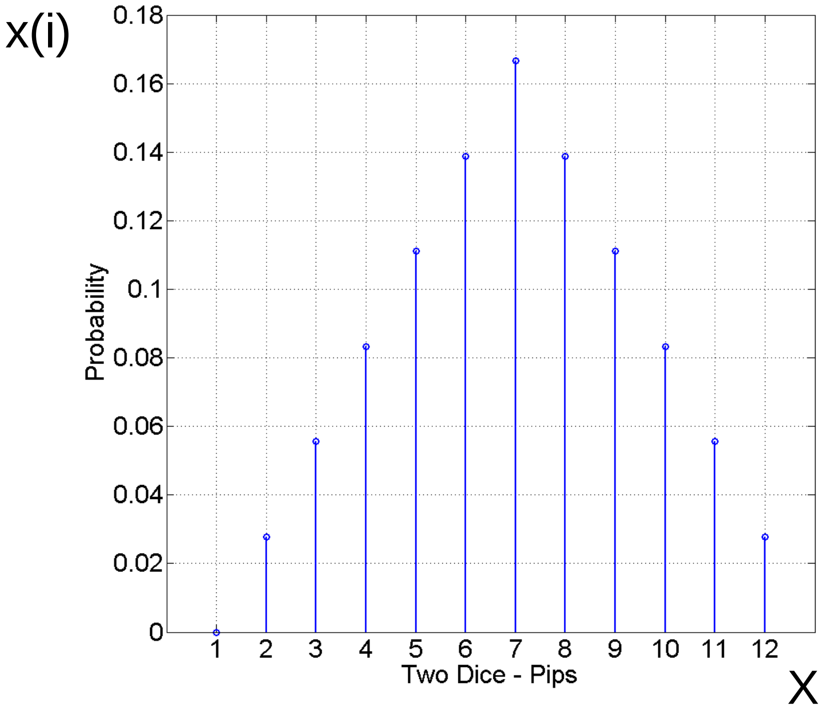
- $t_1 < t_2 \Rightarrow X(t_1) \leq X(t_2)$ (monotony)
- $t_1 < t_2 \Rightarrow P\{t_1 < X \leq t_2\} = X(t_2) - X(t_1)$
- $X(-\infty) = 0 \quad \wedge \quad X(\infty) = 1$
- $X^c(t) = 1 - X(t) = P\{X > t\}$



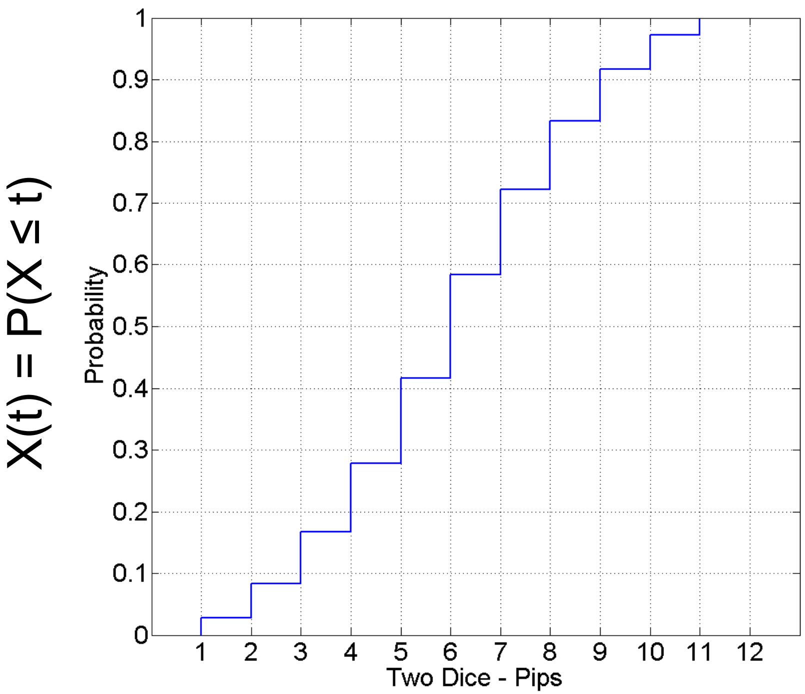
Distribution function

Statistics Fundamentals

Difference between distribution and distribution function

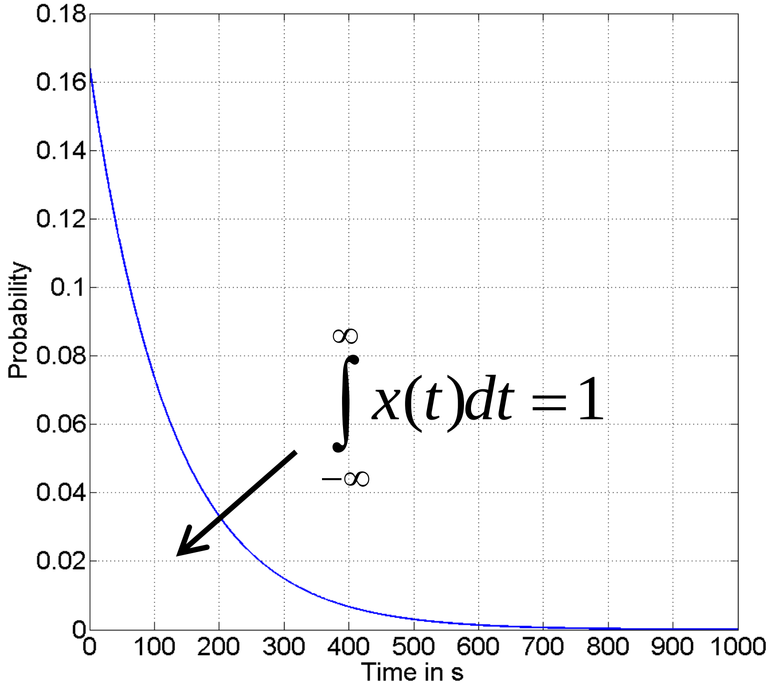


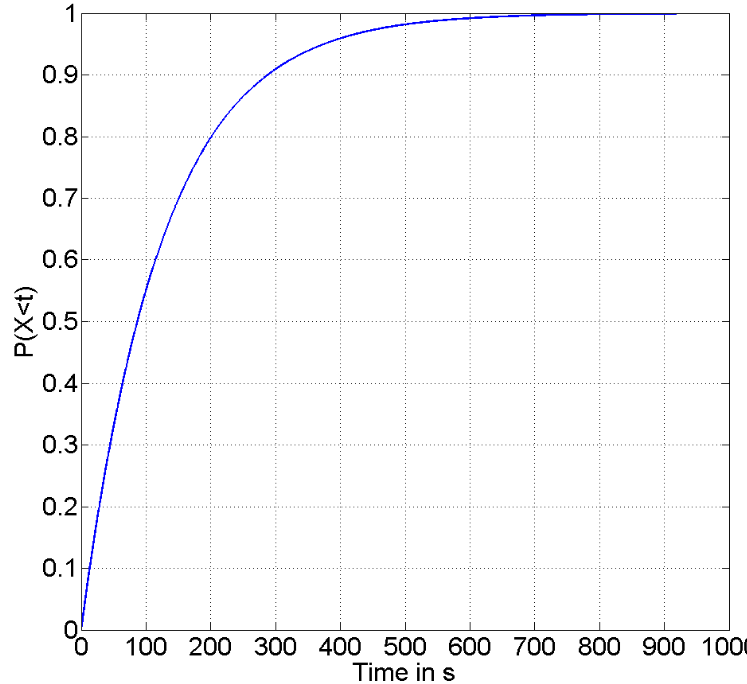
Distribution
(Verteilung)



Distribution function
(Verteilungsfunktion)

Continuous random variable





Probability Density Function (Verteilungsdichtefunktion)

$$x(t) = \frac{d}{dt} X(t)$$

Cumulative Density Function

$$X(t) = \int_{-\infty}^{t} x(t)dt$$

Statistics Fundamentals

- **Expectation** (Erwartungswert)
  - X : Probability density function
  - g(x) : Function of random variable X

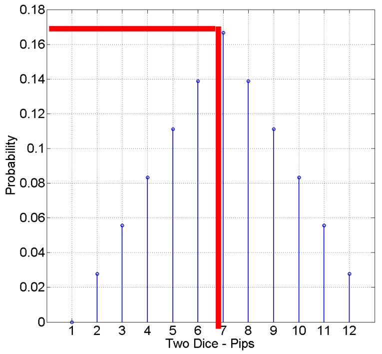
$$E[g(X)] = \int_{-\infty}^{\infty} g(t) \cdot x(t) dt$$

- **Mean** (Mittelwert einer Zufallsvariablen)

$$m_1 = E[X] = \int_{-\infty}^{\infty} t \cdot x(t) dt$$

- **Mode** (Outcome of the random variable with the highest probability)

$$c = Max(x(t))$$



Statistics Fundamentals

- Gewöhnliche Momente einer Zufallsvariablen
  - $g(X) = X^k \Longrightarrow m_k = E\left[X^k\right] = \int_{-\infty}^{\infty} t^k \cdot x(t)dt, \qquad k = 0,1,2,\ldots$

- Central moment (Zentrales Moment)
  - Variation of the random variable in respect to its mean

$$g(X) = (X - m_1)^k$$

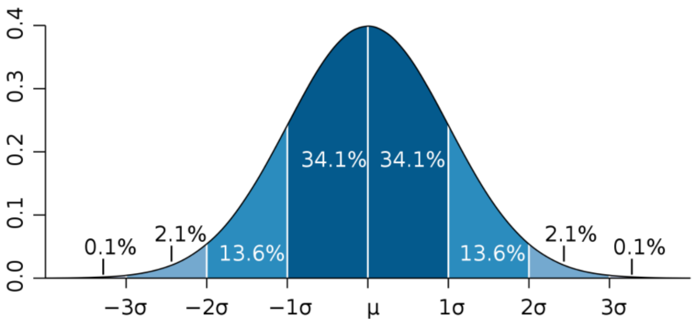
$$\Longrightarrow \mu_k = E\left[(X - m_1)^k\right] = \int_{-\infty}^{\infty} (t - m_1) \cdot x(t)dt, \qquad k = 0,1,2,\ldots$$

  - Special Case (k=2):

$$\mu_2 = E\left[(X - m_1)^2\right] = VAR[X]$$

Statistics Fundamentals

- **Standard deviation** (Standardabweichung)
  - $$\sigma_X = \sqrt{VAR[X]}$$



- **Coefficient of variation** (Variationskoeffizient)
  - $$c_X = \frac{\sigma_X}{E[X]}, \qquad E[X] > 0$$
  - The coefficient of variation is a normalized measure of dispersion of a probability distribution
  - It is a dimensionless number which does not require knowledge of the mean of the distribution in order to describe the distribution

Picture taken from Wikipedia

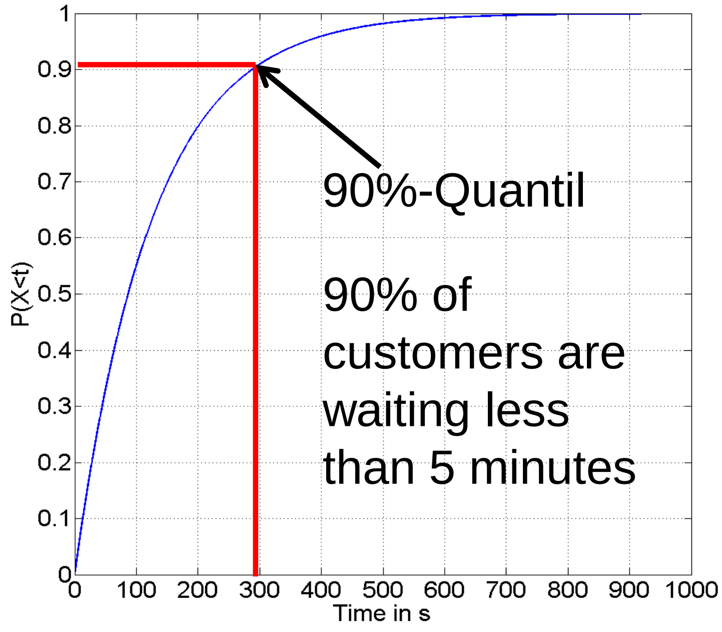
Statistics Fundamentals

- **p-percentile t_p** (p-Quantil)

  A percentile is the value of a variable below which a certain percent of observations fall

  - VDF $F : R \rightarrow (0,1)$ (bijective)
  - $F(x) = P(X < x) = p$
  - $F^{-1}(x) = \inf\{x \in R : p \leq F(x)\}$
  - Special Case:
    - Median 0.5-percentile
    - Upper percentile 0.75-percentile
    - Lower percentile 0.25-percentile
  - Typical Use Case:
    - QoS in networks

      (e.g. 99.9%-percentile of the delay)



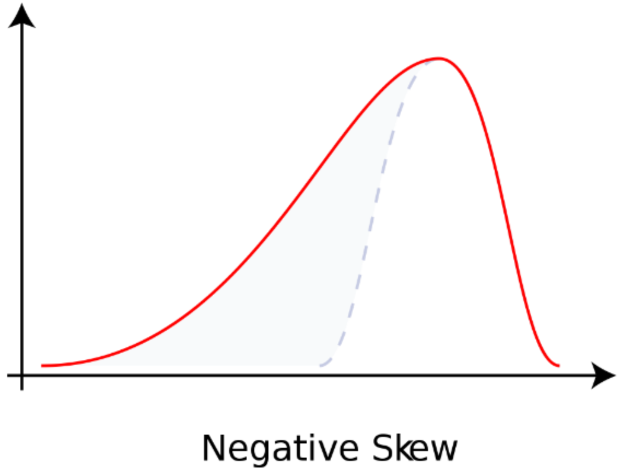
Cumulative Density Function

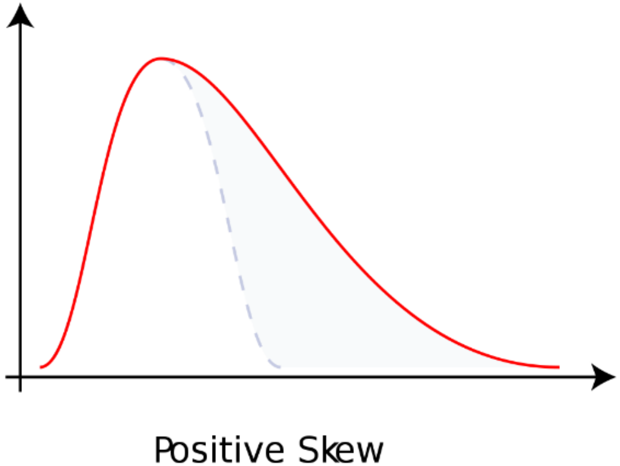
- **Skewness** (Schiefe)

  Skewness describes the asymmetry of a distribution

  - v < 0 : The left tail of the distribution is longer (linksschief)
    => Mass is concentrated in the right
  - v > 0 : The right tail of the distribution is longer (rechtsschief)
    => Mass is concentrated in the left

$$\upsilon_X = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{\mu_3}{\sigma^3}$$





Picture taken from Wikipedia

Statistics Fundamentals

Scalability Issues

- Multiplication of a random variable X with a scalar s
  - $Y = s \cdot X$
  - $E[Y] = s \cdot E[X]$
  - $VAR[Y] = s^2 \cdot VAR[X]$
- Addition of two random variables X and Y
  - $Z = X + Y$
  - $E[Z] = E[X] + E[Y]$
  - $VAR[Z] = VAR[X] + VAR[Y]$ (only if A and B independent)

Covariance

Covariance is a measure which describes how two variables change together

$$Cov(X,Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X] \cdot E[Y]$$

- Special Case: $Cov(X,X) = VAR[X]$
- Other Characteristics:
  - $Cov(X,a) = 0$
  - $Cov(X,Y) = Cov(Y,X)$
  - $Cov(aX,bY) = abCov(X,Y)$
  - $Cov(X+a,Y+b) = Cov(X,Y)$

Correlation function

Correlation function describes how two random variable tend to derivate from their expectation

$$Cor(X,Y) = \frac{Cov(X,Y)}{\sqrt{VAR(X) \cdot VAR(Y)}}$$

- Characteristics:
  - $Y = X$ ⇒ $Cor(X,Y) = 1$ (Maximum positive)
  - $Y = -X$ ⇒ $Cor(X,Y) = -1$ (Maximum negative)
  - $Cor(X,Y) > 0$ Both random variable tend to have either high or low values (difference to their expectation)
  - $Cor(X,Y) < 0$ The random variables differ from each other such that one has high values while the other has low values and vice versa (difference to their expectation)

Statistics Fundamentals

- **Autocorrelation** (LK 4.9)
  - Autocorrelation is the cross-correlation of a signal with itself. In the context of statistics it represents a metric for the similarity between observations of a stochastic process. From a mathematical point of view, autocorrelation can be regarded as a tool for finding repeating patterns of a stochastic process.

Definition:

- Correlation of two samples with distance k from a stochastic process X is given by:

$$\Rightarrow \quad Cor(X,Y) \quad \text{with} \quad Y_i = X_{i+j}$$

Use case:

- Test of random number generators
- Evaluation of simulation results (c.f. Batch-Means)

Statistics Fundamentals

Example:

00101110101001101100010011101010100011

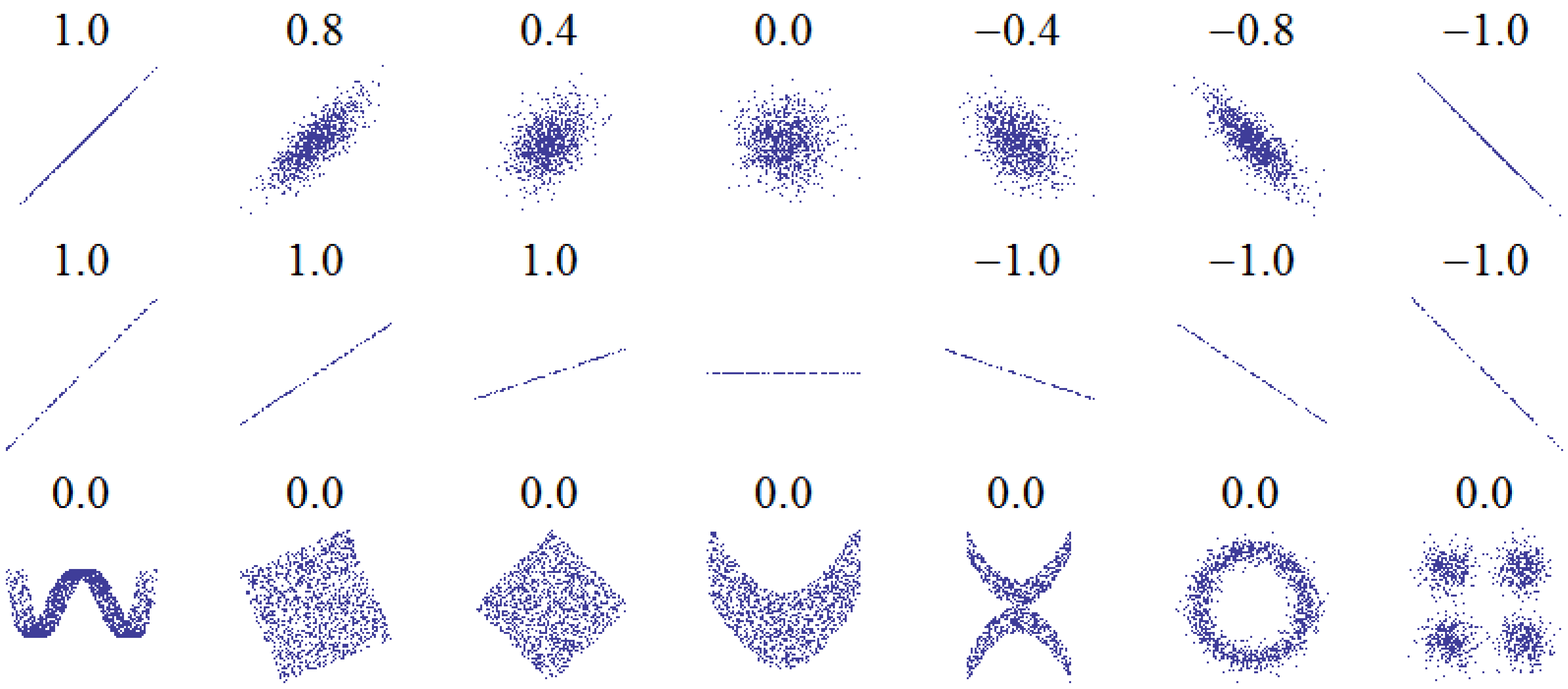
00101110101001101100010011101010100011

Random 

Autocorrelation Lag 4

Visualization of Correlation

Example: Two random variables X and Y are plotted against each other



Picture taken from Wikipedia

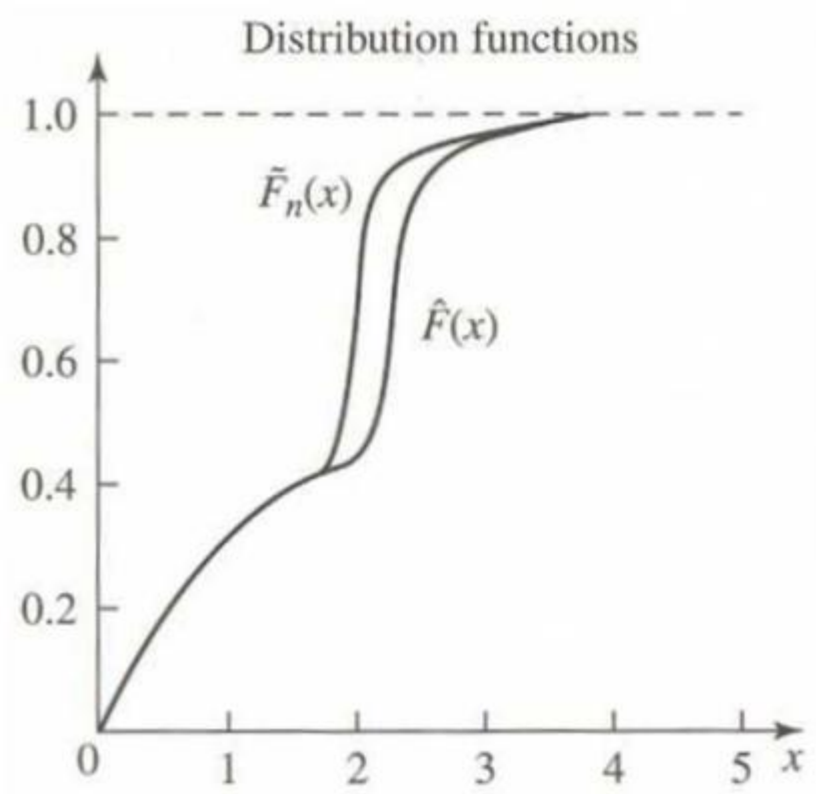
Statistics Fundamentals

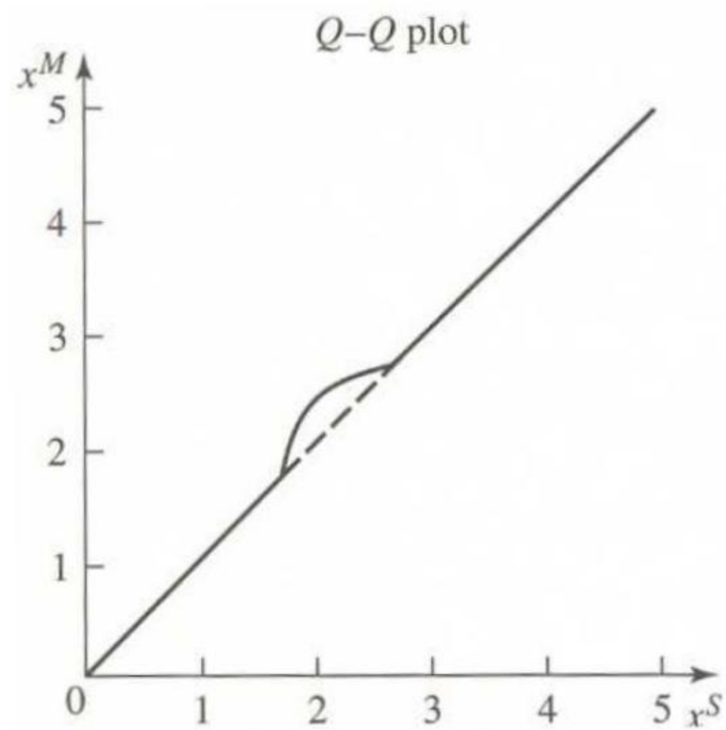
- Visual comparison of different distributions
  - **Quantile-Quantile Plot**
  - **Probability-Probability Plot**

Quantile-Quantile plots (QQ plots)

- Usage: Compare two distributions against each other
  - Usually: Measurement distribution vs. theoretical distribution – do the measurements fit an assumed underlying theoretical model?
  - Also possible: Measurement distribution vs. other measurement distribution – are the two measurement runs really from the same population, or is there variation between the two?
- How it works:
  - Determine 1% quantile, 2% quantile, …, 100% quantile for distributions
  - Plot 1% quantile vs. 1% quantile, 2% quantile vs 2% quantile, etc.
  - Not restricted to percentiles – usually, each of the *n* data points from the measurement is taken as its own 1/*n* quantile
- How to read:
  - If everything is located along the line *x*=*y* then the two distributions are very similar
  - QQ plots amplify discrepancies near the “tail” of the distributions
- Warning about scales:
  - Plot program often automatically assign X and Y different scales
  - Straight line indicates: choice of distribution OK, but parameters don’t fit

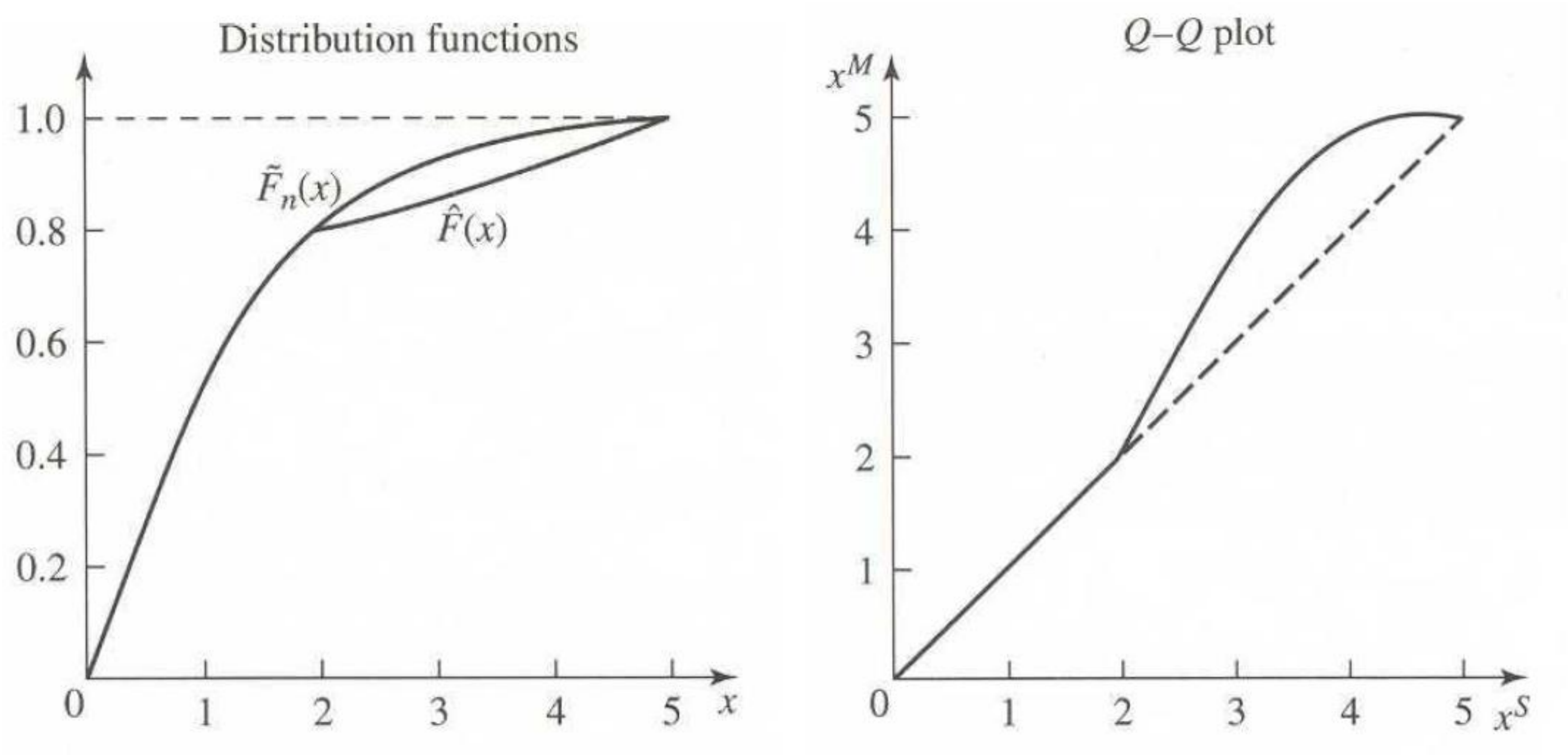
- **QQ Plot**


Distribution functions
1.0
0.8
0.6
0.4
0.2
0
1
2
3
4
5
x
$\tilde{F}_n(x)$
$\hat{F}(x)$


Q–Q plot
x^M
5
4
3
2
1
0
1
2
3
4
5
x^S

Statistics Fundamentals

- QQ Plot



Statistics Fundamentals

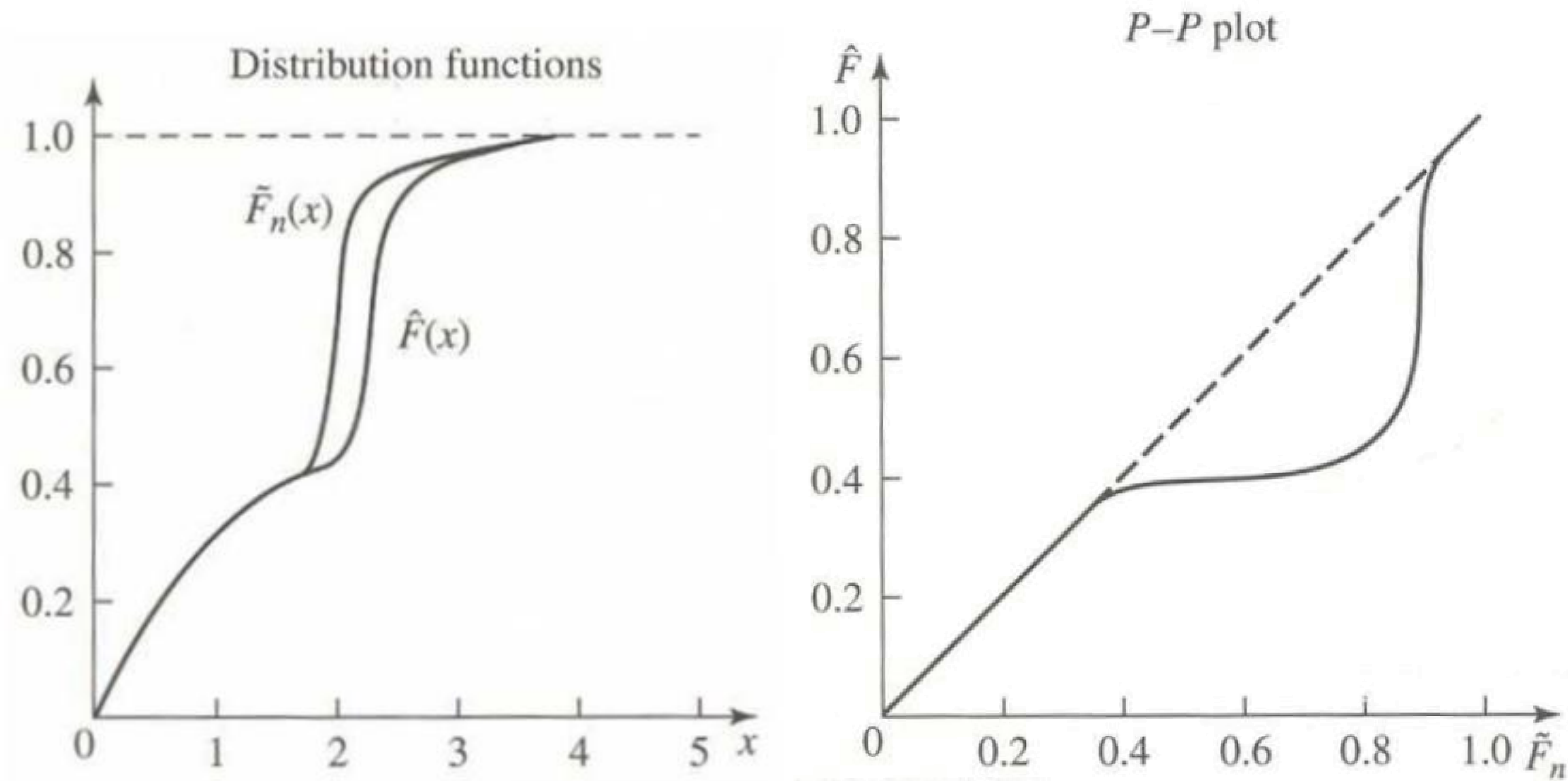
Probability-Probability plots (PP plots)

- Very similar to QQ plot
- QQ plot is more common, though

- Difference to QQ plot:
  - QQ plot compares [quantiles of] two distributions:
    1% quantile vs. 1% quantile, etc.
    - Graphically: the y axes of the cumulative density distribution functions are plotted against each other
  - PP plot compares probabilities of two distributions
    - Graphically: the y axes of the probability density functions are plotted against each other

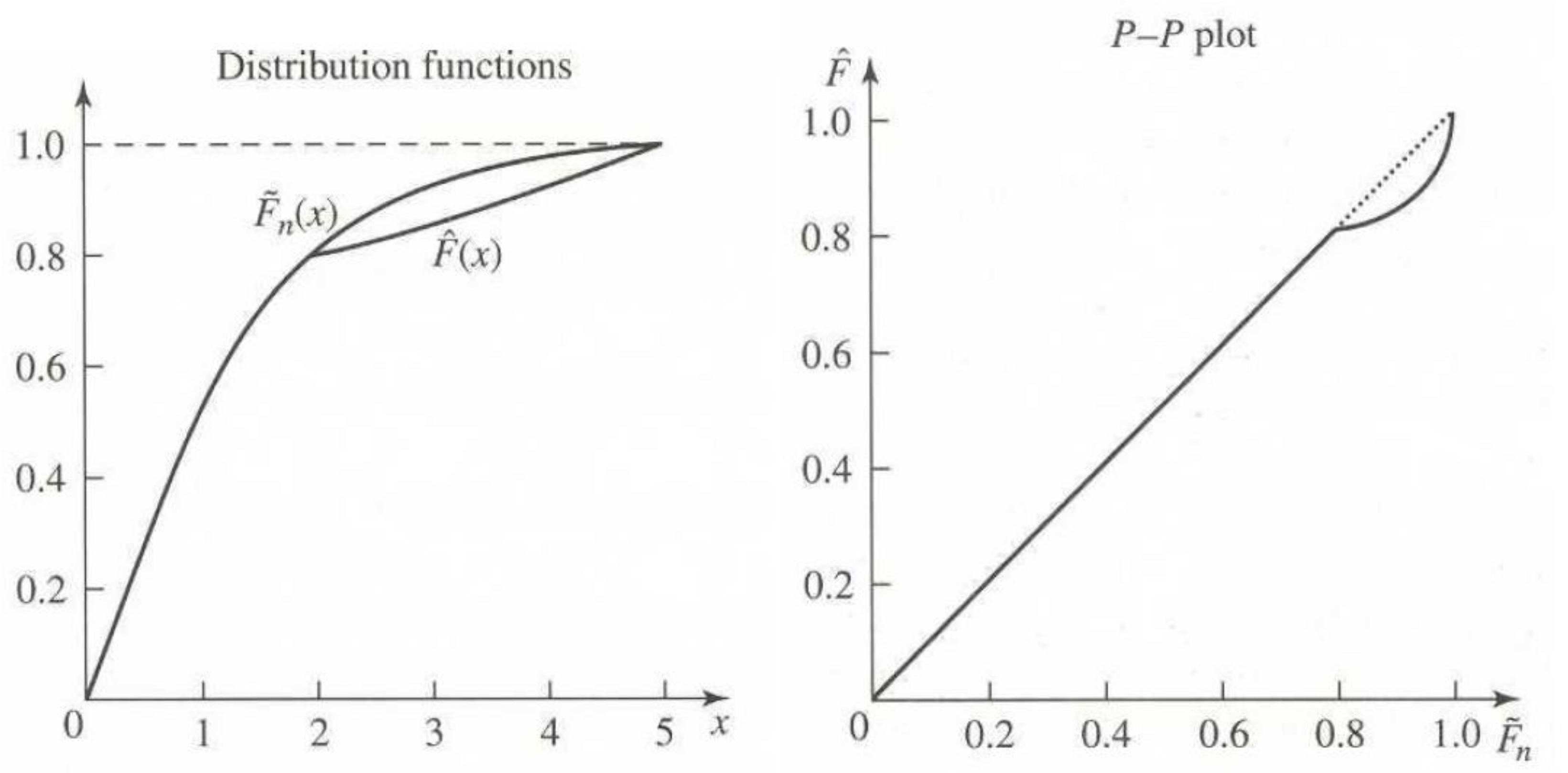
- How to read:
  - Basically the same as QQ plot
  - PP plots highlight differences near the centers of the distributions
    (whereas QQ plots highlight differences near the ends of the distributions)

- **PP Plot**



The difference between the "middles" of $\hat{F}(x)$ and $\tilde{F}_n(x)$ amplified by the *P–P* plot.

- **PP Plot**



The difference between the right tails of $\hat{F}(x)$ and $\tilde{F}_n(x)$ amplified by the Q–Q plot.