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Tutorials for Network Coding (IN3300)
Tutorial 4 – 2014/11/20

Problem 1 Lossy wireless networks

We consider the four-node wireless relay network $G = (N, H)$ depicted in Figure 1 in the lossy hypergraph model with orthogonal MAC. The solution of most subproblems can be written as table (see pre-printed Table 1).

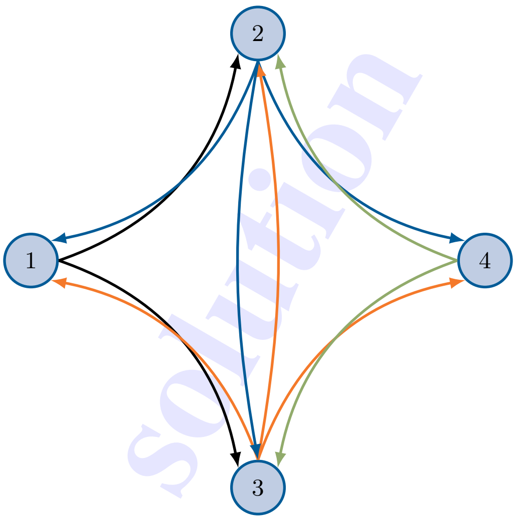


Figure 1: Four-node relay network

a)* Explicitly state the set of hyperarcs H.

See column $(a, B) \in H$ in Table 1.

b) Number the hyperarcs $(a, B) \in H$ in lexicographic ascending order, i.e., $(a, B) < (a', B')$ if

1. $a < a'$ *or*
2. $a = a' \ \wedge \ |B| < |B'|$ *or*
3. $a = a' \ \wedge \ |B| = |B'| \ \wedge \min B < \min B'$,

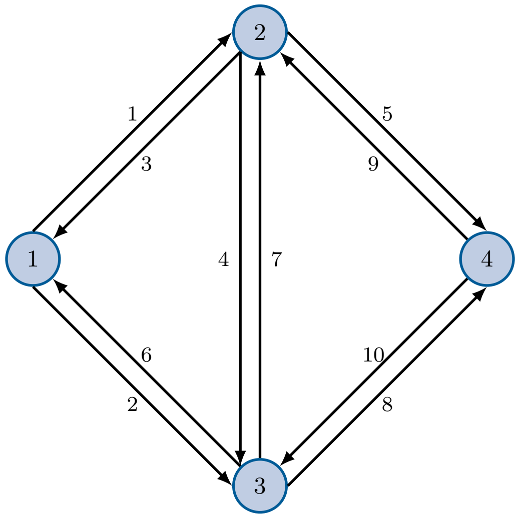
such that $j \equiv (a, B)$ with $j \in \{1, 2, \dots\}$ for all $(a, B) \in H$.

See column $j \equiv (a, B)$ in Table 1.

c)* Explicitly state all arcs $(a, b) \in A$ that are induced by each of the hyperarcs $(a, B) \in H$.

See column (a, b) in Table 1.

d) Draw the graph $G' = (N, A)$ that is induced by G.



e) Number the arcs $(a, b) \in A$ in lexicographic ascending order, i.e., $(a, b) < (a', b')$ if

1. $a < a'$ *or*
2. $a = a' \wedge b < b'$,

such that $k \equiv (a, b)$ with $k \in \{1, 2, \dots\}$ for all $(a, b) \in A$. Also state by which hyperarc $j \equiv (a, B) \in H$ a given arc $k \equiv (a, b) \in A$ is induced by.

$(a, b) \in A$	$k \equiv (a, b)$
$(1, 2)$	1
$(1, 3)$	2
$(2, 1)$	3
$(2, 3)$	4
$(2, 4)$	5
$(3, 1)$	6
$(3, 2)$	7
$(3, 4)$	8
$(4, 2)$	9
$(4, 3)$	10

f) Enumerate the sets A_j for all $j \equiv (a, B) \in H$ such that $(a, b) \equiv k \in A_j$ if hyperarc j induces arch k.

See solution of (c), fourth column.

g) State the hyperarc-arc incidence matrix $\boldsymbol{N}$.

$$
\boldsymbol{N} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}
$$

h) State the hyperarc-hyperarc incidence matrix $\boldsymbol{Q}$.

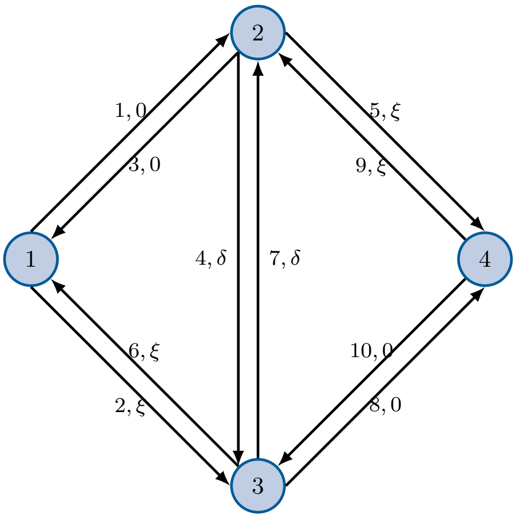
$$
\boldsymbol{M} = \begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}
$$

We now consider a bidirectionally coded session between nodes 1 and 4. Assume that each arch $k \in A$ has unit capacity and a link error probability of $0 \leq \epsilon_k \leq 1$.

i) Determine the hyperarc capacity region $\mathcal{Z}$ assuming that

$$
\begin{aligned}
\tau_1 &= \tau_4 = \tau, \\
\tau_2 &= \tau_3 = \theta, \\
\epsilon_{13} &= \epsilon_{31} = \epsilon_{24} = \epsilon_{42} = \xi, \\
\epsilon_{12} &= \epsilon_{21} = \epsilon_{34} = \epsilon_{43} = 0, \text{ and} \\
\epsilon_{23} &= \epsilon_{32} = \delta.
\end{aligned}
$$

Re-print with of G' with loss probabilities:



See column z_j in Table 1. The capacity region is then given by

$$
\bigcup_{\substack{\tau,\theta \geq 0 \\ \tau + \frac{1}{2}\theta \leq 1}} \{\boldsymbol{z}\}.
$$

j) Determine the broadcast capacity region $\mathcal{Y}$.

See column y_j in Table 1.

k) Enumerate all s–t cuts S and their capacities $v(S_i)$.

S	$v(S)$
$S_1 = \{1\}$	$y_3 = \tau$
$S_2 = \{1,2\}$	$y_2 + y_9 = \tau(1-\xi) + \theta(1-\delta\xi)$
$S_3 = \{1,3\}$	$y_1 + y_{16} = \tau + \theta$
$S_4 = \{1,2,3\}$	$y_6 + y_{13} = \theta(1-\xi) + \theta$
$S_5 = \{4\}$	$y_{20} = \tau$
$S_6 = \{4,2\}$	$y_{19} + y_7 = \tau + \theta$
$S_7 = \{4,3\}$	$y_{18} + y_{14} = \tau(1-\xi) + \theta(1-\delta\xi)$
$S_8 = \{4,3,2\}$	$y_4 + y_{11} = \theta + \theta(1-\xi)$

l) Which cuts are redundant, i.e., which cut can not be the min cut?

The cut S_3 is redundant since $v(S_3) = \tau + \theta \geq \tau(1-\xi) + \theta(1-\delta\xi) = v(S_2)$ for all values of $\delta, \xi \in [0,1]$ and all τ, θ. Similarly, S_6 is redundant.

m) Find the maximum bidirectional communication rate $r = \min(r_1, r_4)$ assuming that $\theta = \frac{1}{2} - \tau$ by computing the min-cut value.

The three potential min-cut values are

$$v(S_1) = v(S_5) = \tau \tag{1}$$

$$v(S_4) = v(S_8) = (\tfrac{1}{2} - \tau)(2-\xi) = \tfrac{1}{2}(2-\xi) - \tau(2-\xi) \tag{2}$$

$$v(S_2) = v(S_7) = \tau(1-\xi) + (\tfrac{1}{2} - \tau)(1-\delta\xi) = \tfrac{1}{2}(1-\delta\xi) - \tau\xi(1-\delta) \tag{3}$$

n) Determine τ such that r is maximized.

The min cut is achieved for $\tau^\star$ defined as the minimum of the intersection points of $v(S_1)$ with $v(S_4)$ and $v(S_2)$ since $v(S_1)$ is increasing and the other two are decreasing. This means that either S_1 and S_4 or S_1 and S_4 are minimum cuts. Therefore, we compute the intersection points $\tau^{(1)}$ and $\tau^{(2)}$ defined as the τ where $v(S_1) = v(S_4)$ and $v(S_1) = v(S_2)$, respectively:

$$\tau^{(1)} = \frac{2-\xi}{6-2\xi} \tag{4}$$

$$\tau^{(2)} = \frac{1-\delta\xi}{2+2\xi(1-\delta)} \tag{5}$$

$$\tag{6}$$

The $\tau^\star$ is given by $\tau^\star = \min\{\tau^{(1)}, \tau^{(2)}\}$ and the min-cut value r is given by the cut value $v(S_1)$ for $\tau = \tau^\star$, i.e.,

$$r = \min\left\{\frac{2-\xi}{6-2\xi}, \frac{1-\delta\xi}{2+2\xi(1-\delta)}\right\}. \tag{7}$$

o) Discuss the extreme cases $\xi \in \{0,1\}$ and $\delta \in \{0,1\}$.

CASE 1 $\xi = 0$:

$$r = \tau^\star = \min\{\tfrac{2}{6}, \tfrac{1}{2}\} = \tfrac{1}{3}$$

There is are two lossless paths from 1 to 4 over nodes 2 and 3. Nodes 1 and 4 get each $\frac{1}{3}$ of the resources (time), nodes 2 and 3 get $\frac{1}{6}$ each.

CASE 2 $\xi = 1$:

$$r = \tau^\star = \min\{\tfrac{1}{4}, \tfrac{1-\delta}{4-2\delta}\} = \frac{1}{2 + \frac{2}{1-\delta}} \leq \frac{1}{4}$$

There is only on path with nonzero capacity between 1 and 4, namely, 1–2–3–4. The link between 2 and 3 is lossy if $\delta > 0$. The higher δ, the more resources are allocated to 2 and 3 and the less are allocated to 1 and 4, i.e., $\tau^\star$ gets smaller the larger δ is.

CASE 3 $\delta = 0$:

$$r = \tau^\star = \min\{\tfrac{2-\xi}{6-2\xi}, \tfrac{1}{2+2\xi}\} = \frac{2-\xi}{6-2\xi} \in [\tfrac{1}{4}, \tfrac{1}{3}]$$

CASE 4 $\delta = 1$:

$$r = \tau^\star = \min\{\frac{2-\xi}{6-2\xi}, \frac{1-\xi}{2}\} \in [0, \frac{1}{3}]$$

$(a, B) \in H$	$j \equiv (a, B)$	(a, b)	A_j	z_j	y_j
$(1, \{2\})$	1	$(1, 2)$	$\{1\}$	$\tau\xi$	τ
$(1, \{3\})$	2	$(1, 3)$	$\{2\}$	0	$\tau(1 - \xi)$
$(1, \{2, 3\})$	3	$(1, 2), (1, 3)$	$\{1, 2\}$	$\tau(1 - \xi)$	τ
$(2, \{1\})$	4	$(2, 1)$	$\{3\}$	$\theta\delta\xi$	θ
$(2, \{3\})$	5	$(2, 3)$	$\{4\}$	0	$\theta(1 - \delta)$
$(2, \{4\})$	6	$(2, 4)$	$\{5\}$	0	$\theta(1 - \xi)$
$(2, \{1, 3\})$	7	$(2, 1), (2, 3)$	$\{3, 4\}$	$\theta(1 - \delta)\xi$	θ
$(2, \{1, 4\})$	8	$(2, 1), (2, 4)$	$\{3, 5\}$	$\theta\delta(1 - \xi)$	θ
$(2, \{3, 4\})$	9	$(2, 3), (2, 4)$	$\{4, 5\}$	0	$\theta(1 - \delta\xi)$
$(2, \{1, 3, 4\})$	10	$(2, 1), (2, 3), (2, 4)$	$\{3, 4, 5\}$	$\theta(1 - \delta)(1 - \xi)$	θ
$(3, \{1\})$	11	$(3, 1)$	$\{6\}$	0	$\theta(1 - \xi)$
$(3, \{2\})$	12	$(3, 2)$	$\{7\}$	0	$\theta(1 - \delta)$
$(3, \{4\})$	13	$(3, 4)$	$\{8\}$	$\theta\delta\xi$	θ
$(3, \{1, 2\})$	14	$(3, 1), (3, 2)$	$\{6, 7\}$	0	$\theta(1 - \xi\delta)$
$(3, \{1, 4\})$	15	$(3, 1), (3, 4)$	$\{6, 8\}$	$\theta(1 - \xi)\delta$	θ
$(3, \{2, 4\})$	16	$(3, 2), (3, 4)$	$\{7, 8\}$	$\theta(1 - \delta)\xi$	θ
$(3, \{1, 2, 4\})$	17	$(3, 1), (3, 2), (3, 4)$	$\{6, 7, 8\}$	$\theta(1 - \delta)(1 - \xi)$	θ
$(4, \{2\})$	18	$(4, 2)$	$\{9\}$	0	$\tau(1 - \xi)$
$(4, \{3\})$	19	$(4, 3)$	$\{10\}$	$\tau\xi$	τ
$(4, \{2, 3\})$	20	$(4, 2), (4, 3)$	$\{9, 10\}$	$\tau(1 - \xi)$	τ

Table 1: Fill in values from different subproblems.