Technische Universität München
Lehrstuhl Informatik VIII
Prof. Dr.-Ing. Georg Carle
Prof. Dr.-Ing. Wolfgang Utschick
Stephan M. Günther
Maximilian Riemensberger

Tutorials for Network Coding (IN3300)

Tutorial 2 – 2014/10/30

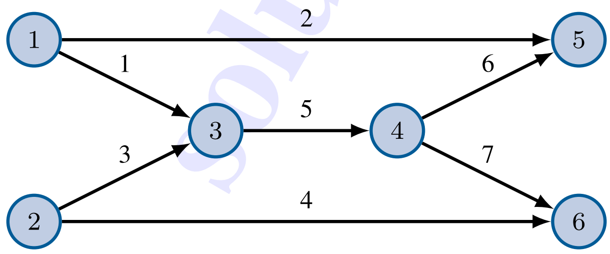
Problem 1 Maximum flow problem

We consider the wired network with $n = 6$ nodes and $m = 7$ arcs that is described by the incidence matrix

$$\boldsymbol{M} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}.$$

Arcs are enumerated in lexicographic order as known from the lecture, e.g. $(1, 2) < (2, 1)$.

a)* Draw the network described by $\boldsymbol{M}$ and label both nodes and arcs.



b)* What is the rank $\boldsymbol{M}$?

rank $\boldsymbol{M} = n - 1 = 5$ since one row is linear dependent.

c)* Determine a basis B of null $\boldsymbol{M}$.

$$B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The arc capacities are given by $\boldsymbol{z} = [2\ 2\ 2\ 2\ 1\ 2\ 2]$. We consider a single unicast between nodes 1 and 6 described by $\boldsymbol{d} = [1\ 0\ 0\ 0\ 0\ \ -1]$.

d)* Determine the capacity between nodes 1 and 6 using the min-cut / max-flow theorem. (A bit tedious to enumerate all the cuts . . .)

Set containing s	**Set containing t**	**cut capacity**
$\{1\}$	$\{2,3,4,5,6\}$	$2+2=4$
$\{1,3\}$	$\{2,4,5,6\}$	$2+1=3$
$\{1,5\}$	$\{2,3,4,6\}$	2
$\{1,3,4\}$	$\{2,5,6\}$	$2+2+2=6$
$\{1,3,5\}$	$\{2,4,6\}$	1
$\{1,3,4,5\}$	$\{2,6\}$	2
...	...	...

The maximum value for flow $(1,6)$ is the minimum of all cut capacities and thus 1.

The maximum flow problem is formally expressed as linear program

$$\max_{r,\boldsymbol{x}}\ r \quad \text{s.t.} \quad \boldsymbol{M}\boldsymbol{x} = r\boldsymbol{d}, \tag{1}$$
$$\boldsymbol{x} \geq \boldsymbol{0}, \tag{2}$$
$$\boldsymbol{x} \leq \boldsymbol{z}. \tag{3}$$

The demand vector $\boldsymbol{d}$ is chosen such that its elements are zero except for $d_s = 1$ and $d_t = -1$. In order to solve this problem using Matlab we have to rewrite it as

$$\min_{\boldsymbol{x}}\ \boldsymbol{f}^T\boldsymbol{x} \quad \text{s.t.} \quad \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{a}, \tag{4}$$
$$\boldsymbol{B}\boldsymbol{x} = \boldsymbol{b}, \tag{5}$$
$$\boldsymbol{x} \geq 0, \tag{6}$$
$$\boldsymbol{x} \leq \boldsymbol{z}. \tag{7}$$

e)* Express the scalar rate r by means of $\boldsymbol{M}$, $\boldsymbol{x}$, and $\boldsymbol{d}$.

Multiplying both sides of (1) by $\boldsymbol{d}^T/(\boldsymbol{d}^T\boldsymbol{d})$ gives

$$\begin{aligned}\boldsymbol{M}\boldsymbol{x} &= r\boldsymbol{d},\\ \frac{\boldsymbol{d}^T\boldsymbol{M}\boldsymbol{x}}{\boldsymbol{d}^T\boldsymbol{d}} &= r\frac{\boldsymbol{d}^T\boldsymbol{d}}{\boldsymbol{d}^T\boldsymbol{d}},\\ r &= \frac{\boldsymbol{d}^T\boldsymbol{M}\boldsymbol{x}}{\boldsymbol{d}^T\boldsymbol{d}}.\end{aligned}$$

f) Determine $\boldsymbol{B}$ such that $\boldsymbol{B}\boldsymbol{x} = \boldsymbol{0}$ is equivalent to $\boldsymbol{M}\boldsymbol{x} = r\boldsymbol{d}$.

Using the result for r we obtain:

$$\boldsymbol{M}\boldsymbol{x} = r\boldsymbol{d}$$
$$\boldsymbol{M}\boldsymbol{x} = \frac{\boldsymbol{d}^T\boldsymbol{M}\boldsymbol{x}}{\boldsymbol{d}^T\boldsymbol{d}}\boldsymbol{d}$$

Since $\boldsymbol{d}^T\boldsymbol{M}\boldsymbol{x}$ is a scalar value, we can rearrange the equation:

$$\boldsymbol{M}\boldsymbol{x} = \boldsymbol{d}\frac{\boldsymbol{d}^T\boldsymbol{M}\boldsymbol{x}}{\boldsymbol{d}^T\boldsymbol{d}}$$
$$\boldsymbol{M}\boldsymbol{x} - \boldsymbol{d}\frac{\boldsymbol{d}^T\boldsymbol{M}\boldsymbol{x}}{\boldsymbol{d}^T\boldsymbol{d}} = \boldsymbol{0}$$
$$\left(\boldsymbol{I} - \frac{\boldsymbol{d}\boldsymbol{d}^T}{\boldsymbol{d}^T\boldsymbol{d}}\right)\boldsymbol{M}\boldsymbol{x} = \boldsymbol{0}$$
$$\Rightarrow \boldsymbol{B} = \left(\boldsymbol{I} - \frac{\boldsymbol{d}\boldsymbol{d}^T}{\boldsymbol{d}^T\boldsymbol{d}}\right)\boldsymbol{M}$$

g) Determine $\boldsymbol{f}$ such that $\boldsymbol{f}^T\boldsymbol{x} = r$.

$$\boldsymbol{f}^T\boldsymbol{x} = r$$
$$\boldsymbol{f}^T\boldsymbol{x} = \frac{\boldsymbol{d}^T\boldsymbol{M}}{\boldsymbol{d}^T\boldsymbol{d}}\boldsymbol{x}$$
$$\Rightarrow \boldsymbol{f} = \frac{\boldsymbol{M}^T\boldsymbol{d}}{\boldsymbol{d}^T\boldsymbol{d}}$$

h) State the revised optimization problem and solve it using Matlab.

$$\min_{\boldsymbol{x}} \; -\boldsymbol{f}^T\boldsymbol{x} \quad \text{s.t.} \quad \boldsymbol{B}\boldsymbol{x} = \boldsymbol{0}, \quad \boldsymbol{x} \geq 0, \quad \boldsymbol{x} \leq \boldsymbol{z}.$$

An alternative solution to e) – h)

We consider an s-t flow problem so that $d_s = 1$, $d_t = -1$, and $d_i = 0$ for all $i \neq s, t$. Let $\boldsymbol{m}_s^{\mathrm{T}}$ and $\boldsymbol{m}_t^{\mathrm{T}}$ denote the rows of the incidence matrix $\boldsymbol{M}$ corresponding to nodes s and t, respectively, and let $\boldsymbol{M}_{st}$ denote the matrix $\boldsymbol{M}$ with those two rows removed. For example if $s = 1$ and $t = 6$, then $\boldsymbol{M}_{st}$ is the matrix $\boldsymbol{M}$ with its first and last row removed.
Since The matrix $\boldsymbol{1}^{\mathrm{T}}\boldsymbol{M} = \boldsymbol{0}^{\mathrm{T}}$ and $\boldsymbol{1}^{\mathrm{T}}\boldsymbol{d} = 0$, one of the $n-1$ constraints comprised by $\boldsymbol{M}\boldsymbol{x} = r\boldsymbol{d}$ is redundant and can be dropped. We choose to drop the flow conservation constraint at node t, i.e., we remove the row $\boldsymbol{m}_t^{\mathrm{T}}$ from $\boldsymbol{M}$ and the element $d_t = -1$ from $\boldsymbol{d}$. Furthermore, since s is the unique source node in this problem,

only the flow conservation constraint at node s involves r. That is,

$$\boldsymbol{m}_s^{\mathrm{T}}\boldsymbol{x} = rd_s = r$$

since $d_s = 1$. Therefore, we can substitute the objective function of the maximization r by $\boldsymbol{m}_s^{\mathrm{T}}\boldsymbol{x}$ and remove the flow conservation constraint at node s from the set of constraints. The remaining flow conservation constraints independent of r and given by

$$\boldsymbol{M}_{st}\boldsymbol{x} = \boldsymbol{0}.$$

This yields the problem

$$\max_{\boldsymbol{x}} \ \boldsymbol{m}_s^{\mathrm{T}}\boldsymbol{x} \quad \text{s.t.} \quad \begin{aligned} \boldsymbol{M}_{st}\boldsymbol{x} &= \boldsymbol{0}, \\ \boldsymbol{x} &\geq \boldsymbol{0}, \\ \boldsymbol{x} &\leq \boldsymbol{z}. \end{aligned}$$

This problem can be equivalently posed as the minimization problem

$$\min_{\boldsymbol{x}} \ -\boldsymbol{m}_s^{\mathrm{T}}\boldsymbol{x} \quad \text{s.t.} \quad \begin{aligned} \boldsymbol{M}_{st}\boldsymbol{x} &= \boldsymbol{0}, \\ \boldsymbol{x} &\geq \boldsymbol{0}, \\ \boldsymbol{x} &\leq \boldsymbol{z}. \end{aligned}$$

This problem is in a form suitable for Matlab and has the significant advantage (from the perspective of numerical stability) that $\boldsymbol{M}_{st}$ has full row rank, which is not the case with $\boldsymbol{M}$ and in particular $(\boldsymbol{I} - \frac{1}{\|\boldsymbol{d}\|^2}\boldsymbol{d}\boldsymbol{d}^{\mathrm{T}})\boldsymbol{M}$.

Now we consider the case of a second flow, i. e., we have two demand vectors

$$\begin{aligned} \boldsymbol{d}_1 &= [1\,0\,0\,0\,0\ -1]^T \text{ and} \\ \boldsymbol{d}_2 &= [0\,1\,0\,0\ -1\,0]^T. \end{aligned}$$

If we want to maximize the joint rate $r = r_1 + r_2$, the optimization problem becomes

$$\max_{r_1, r_2} \ r_1 + r_2 \quad \text{s.t.} \quad \begin{aligned} \boldsymbol{M}\boldsymbol{x}_1 &= r_1\boldsymbol{d}_1, \\ \boldsymbol{M}\boldsymbol{x}_2 &= r_2\boldsymbol{d}_2, \\ \boldsymbol{x}_1, \boldsymbol{x}_2 &\geq \boldsymbol{0}, \\ \boldsymbol{x}_1 + \boldsymbol{x}_2 &\leq \boldsymbol{z}. \end{aligned}$$

We now restate this problem to solve it in Matlab. To this end, we define

$$\boldsymbol{N} = \begin{bmatrix} \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M} \end{bmatrix}, \ \boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix}, \ \boldsymbol{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \text{ and } \boldsymbol{D} = \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{d}_2 \end{bmatrix}.$$

i) Express $\boldsymbol{r}$ by $\boldsymbol{N}$, $\boldsymbol{D}$, and $\boldsymbol{x}$.

Using the definitions above, the equality constraint in (1) now reads as

$$\boldsymbol{N}\boldsymbol{x} = \boldsymbol{D}\boldsymbol{r}.$$

Note that $\boldsymbol{D}^T\boldsymbol{D}$ gives a 2×2 matrix where only the diagonal elements are non-zero. Therefore, $\left(\boldsymbol{D}^T\boldsymbol{D}\right)^{-1}$

exists and we can determine $\boldsymbol{r}$ as follows:

$$\begin{aligned}
\boldsymbol{N}\boldsymbol{x} &= \boldsymbol{D}\boldsymbol{r} \\
\boldsymbol{D}^T\boldsymbol{N}\boldsymbol{x} &= \boldsymbol{D}^T\boldsymbol{D}\boldsymbol{x} \\
\boldsymbol{r} &= \left(\boldsymbol{D}^T\boldsymbol{D}\right)^{-1}\boldsymbol{D}^T\boldsymbol{N}\boldsymbol{x}
\end{aligned}$$

j) Determine $\boldsymbol{B}$ such that $\boldsymbol{B}\boldsymbol{x} = \boldsymbol{0}$ is equivalent to $\boldsymbol{N}\boldsymbol{x} = \boldsymbol{D}\boldsymbol{r}$.

Using the result for $\boldsymbol{r}$ we obtain:

$$\begin{aligned}
\boldsymbol{N}\boldsymbol{x} &= \boldsymbol{D}\boldsymbol{r} \\
\boldsymbol{N}\boldsymbol{x} &= \boldsymbol{D}\left(\boldsymbol{D}^T\boldsymbol{D}\right)^{-1}\boldsymbol{D}^T\boldsymbol{N}\boldsymbol{x} \\
\boldsymbol{N}\boldsymbol{x} - \boldsymbol{D}\left(\boldsymbol{D}^T\boldsymbol{D}\right)^{-1}\boldsymbol{D}^T\boldsymbol{N}\boldsymbol{x} &= \boldsymbol{0} \\
\left(\boldsymbol{I} - \boldsymbol{D}\left(\boldsymbol{D}^T\boldsymbol{D}\right)^{-1}\boldsymbol{D}^T\right)\boldsymbol{N}\boldsymbol{x} &= \boldsymbol{0} \\
\Rightarrow \boldsymbol{B} &= \boldsymbol{I} - \boldsymbol{D}\left(\boldsymbol{D}^T\boldsymbol{D}\right)^{-1}\boldsymbol{D}^T
\end{aligned}$$

k) Determine $\boldsymbol{A}$ such that $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{z}$ describes the joint capacity constraint.

We are looking for $\boldsymbol{A}$ such that

$$\boldsymbol{A}\begin{bmatrix} x_{11} \\ \vdots \\ x_{17} \\ x_{21} \\ \vdots \\ x_{27} \end{bmatrix} = \begin{bmatrix} x_{11} + x_{21} \\ \vdots \\ x_{17} + x_{27} \end{bmatrix},$$

and therefore $\boldsymbol{A} = [\boldsymbol{I}_7 \; \boldsymbol{I}_7]$ where $\boldsymbol{I}_7$ denotes a 7×7 unit matrix.

l) Determine $\boldsymbol{f}$ such that $\boldsymbol{f}^T\boldsymbol{x} = r_1 + r_2$.

$$\begin{aligned}
r_1 + r_2 &= \boldsymbol{1}^T\boldsymbol{r} \\
&= \boldsymbol{1}^T\left(\boldsymbol{D}^T\boldsymbol{D}\right)^{-1}\boldsymbol{D}^T\boldsymbol{N}\boldsymbol{x} \\
\Rightarrow \boldsymbol{f}^T &= \boldsymbol{1}^T\left(\boldsymbol{D}^T\boldsymbol{D}\right)^{-1}\boldsymbol{D}^T\boldsymbol{N} \\
\Rightarrow \boldsymbol{f} &= \boldsymbol{N}^T\boldsymbol{D}\left(\boldsymbol{D}^T\boldsymbol{D}\right)^{-T}\boldsymbol{1}
\end{aligned}$$

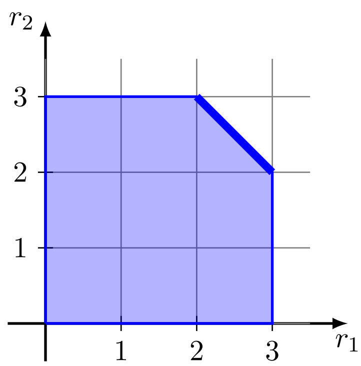
m) State the final problem and solve it in Matlab.

$$\min_{\boldsymbol{x}} \; -\boldsymbol{f}^T\boldsymbol{x} \quad \text{s.t.} \quad \boldsymbol{B}\boldsymbol{x} = \boldsymbol{0}, \\ \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{z}, \\ \boldsymbol{x} \geq \boldsymbol{0}.$$

A Matlab script that finds an optimal solution to that problem is found in the git repository. Note that the script finds just one of an infinite number of possible solutions since the optimization problem does not specify how the sum rate r should be achieved. There is no constraint that demands an equal contribution of r_1 and r_2.

n) Sketch the achievable rate region.

The blue region marks the feasible set of solutions (r_1, r_2). The thick blue line marks the set of optimal solutions that maximize $r = r_1 + r_2$:



Each transmitter (nodes 1 and 2) may transmit at rates $r_i = 2$ without having any influence on the transmit rate of the other source, i.e., without utilizing the shared link $(3, 4)$. Without loss of generality let node 1 fully utilize that shared link, which allows to transmit at rate $r_1 = 3$. The second node may still transmit at rate $r_2 = 2$. For $2 < r_1 \leq 3$ the residual capacity on the shared link may be utilized by node 2 such that $r_1 + r_2 = 5$.

The solution found by Matlab is some point along the line of optimal solutions, i.e., any rate vector $\boldsymbol{x}$ such that $r_1 + r_2 = 5$.