An Efficient Waterfilling Algorithm for Multiple Access OFDM

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Abstract – **In this paper we present a novel and computationally efficient waterfilling algorithm for multiuser OFDM. This algorithm is based on the multiuser waterfilling theorem and determines the subcarrier allocation for a multiple access OFDM system. This approach maximizes the total bitrate under the constraints of user-individual power budgets. Once the subcarrier allocation has been established, the bit and power allocation for each user can be determined with a single-user bitloading algorithm, for which several implementations can be found in the literature. The presented iterative algorithm performs well, also with a high number of subchannels.**

I. Introduction

Recently, there has been a growing interest in broadband multiple access systems like wireless local area networks, cellular mobile communication systems and bidirectional hybrid fiber coax (HFC) networks. Owing to their capability to combat multipath fading and frequency-selective interference, multicarrier systems have found their way into many wireless and wireline applications such as DAB, DVB-T, HiperLAN/2, ADSL and they are discussed for future use in 4G mobile communication.

The theoretical solution for achieving the channel capacity for spectrally-shaped channels with Gaussian noise has been found quite some time ago [1] and in the last decade several practical implementations have been presented (see [2] and the references therein). For multiple access channels, the information theoretical foundations have been laid by Cheng and Verdú [3], who studied the capacity regions for multiple access channels and found a generalization of the single-user waterfilling theorem.

In this paper, we consider an algorithm which applies the multiuser waterfilling theorem to orthogonal frequency division multiple access (OFDMA) systems. A typical example for a multicarrier multiple access system is the uplink in a wireless LAN or in a mobile communication system, but the same technique can be applied to wireline systems such as the upstream in a bidirectional HFC network. In such environments, each user has its own transfer function and transmit power constraint. If the channel is known to the transmitter and the receiver, it can be shown that OFDMA with adaptive subcarrier allocation and adaptive modulation is superior to other multiuser techniques like TDMA or CDMA. This is intuitively clear, as CDMA and TDMA do not make much use of channel state information (CSI) and are not adapted to a specific channel but are mostly used in systems where no CSI is available. Although adaptive bitloading only works perfect for time-invariant channels, it will offer benefits for slowly time-varying channels, too – especially when high spectral efficiency is demanded and fast channel estimation is available.

For single-user OFDM, a bitloading algorithm determines for each subcarrier the power and the number of bits per QAM symbol. For OFDMA, first the subcarrier allocation algorithm assigns the subcarriers to the users, then a bitloading algorithm determines the power and number of bits on each subcarrier. Thus, for the second step the same algorithms as for single-user OFDM can be employed. As these algorithms have reached a state of maturity and several implementations can be found in the literature [2], we do not consider single-user bitloading in this paper. Instead, we concentrate on the subcarrier allocation based on the multiuser waterfilling theorem. This maximizes the total bitrate under the constraint of a maximum transmit power per user. An algorithm which computes the power spectral density for each user has been presented recently [4]. Other approaches minimize e.g. the total transmit power while guaranteeing a minimum bitrate for each user [5].

II. System Model

We consider a multiple access OFDM system where the channel is known to transmitter and receiver. This is usually the case for a bidirectional transmission system where CSI is available at the receiver side after channel estimation and a signalling channel can be used to pass the CSI to the transmitter. We assume a linear time-invariant channel with intersymbol interference (ISI) and additive Gaussian noise. Provided that the length of the cyclic prefix is chosen longer than the longest impulse response, the channel can be decomposed into N independent flat subchannels with channel gain coefficients $H_{u,n}$ for user u and subchannel n, as illustrated in Fig. 1. The noise sequences $r_n[k]$ are assumed to be independent, white Gaussian with zero mean and variances $\sigma_n^2 = E[|r_n[k]|^2]$, where $E[\cdot]$ denotes the expectation operator. Note, as the variances are not necessarily identical,

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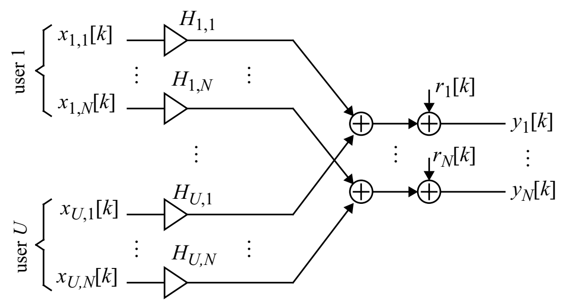

Fig. 1. Channel model for multiuser OFDM.

the assumed noise on the broadband channel is not restricted to be white. The input sequences $x_{u,n}[k]$ are QAM-modulated symbols with mean energy per symbol[1] $E_{u,n} = E[|x_{u,n}[k]|^2]$ and $b_{u,n}$ bits per symbol, with

$$b_{u,n} \in B = \{0, 1, \ldots, b_{max}\} .$$

According to [6], the necessary symbol energy in order to transmit b bits with symbol error probability $P_{S,u}$ is given by

$$E_{u,n} = \frac{\Gamma_u \cdot \sigma_n^2}{|H_{u,n}|^2} \cdot (2^b - 1) \tag{1}$$

where Γ_u is the SNR gap defined by

$$\Gamma_u = \frac{1}{3} \cdot \left[Q^{-1}\left(\frac{P_{S,u}}{4}\right)\right]^2 , \Gamma_u \geq 1 \tag{2}$$

and $Q^{-1}(\cdot)$ is the inverse Q-function. A coding gain can be considered in Γ_u, thus allowing different users to employ different coding schemes [6].

Solving (1) for b and setting $\Gamma_u = 1$ gives Shannon's famous formula for the channel capacity. Thus, the SNR gap Γ_u can be interpreted as a link between the information theoretic channel capacity and the bitrate which is achievable with (coded) QAM modulation. Setting Γ_u according to (2) is hence the first step of applying the waterfilling theorem to multicarrier systems with QAM modulation.

Before delving into the details of the multiuser waterfilling theorem, we revisit the basics of the single-user case.

A. Single-user Waterfilling

With regard to the waterfilling theorem, the channel is completely characterized by its channel gain to noise ratio (CNR), defined as

$$T_n = \frac{|H_n|^2}{\Gamma \cdot \sigma_n^2}, n = 1, \ldots, N . \tag{3}$$

1. Note that this corresponds to the average transmit power of user u on subcarrier n.

In this equation, we already incorporated the SNR gap [6], which accounts for the desired symbol error ratio (SER), assuming QAM modulation. For the classical waterfilling theorem which maximizes the channel capacity, $\Gamma = 1$ has to be chosen.

Then, according to the waterfilling theorem, the power on subchannel n is given by

$$E_n = [c_0 - T_n^{-1}]^+ \text{, where } [x]^+ = \begin{cases} x & \text{for } x > 0 \\ 0 & \text{else} \end{cases} \tag{4}$$

and the "water level" c_0 must be chosen such that

$$E_{tot} = \sum_{n=1}^{N} E_n . \tag{5}$$

Eq. (4), (5) can be visualized with the waterfilling diagram shown in Fig. 2(a).

B. Multiuser Waterfilling

In the multiple access channel shown in Fig. 1, the received symbol sequence on subchannel n is given by

$$y_n[k] = \sum_{u=1}^{U} H_{u,n} \cdot x_{u,n}[k] + r_n[k] \tag{6}$$

and the CNRs are defined by

$$T_{u,n} = \frac{|H_{u,n}|^2}{\Gamma_u \cdot \sigma_n^2} . \tag{7}$$

The generalization of the waterfilling theorem involves the idea of an equivalent channel $\hat{H}_{u,n} = H_{u,n}/\sqrt{b_u}$, which leads to the adoption of the equivalent transmit power $\hat{E}_{u,n} = b_u \cdot E_{u,n}$. This approach makes it possible to combine

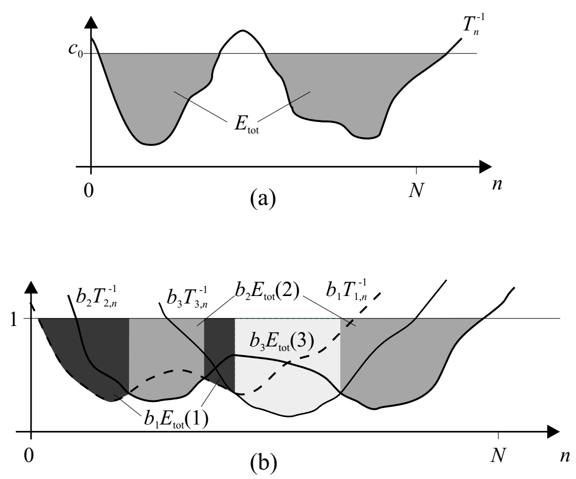

Fig. 2. (a) Single-user waterfilling diagram, (b) Multiuser waterfilling diagram.

the waterfilling diagrams of different users: for each user, the multiplier b_u is chosen such that the water level is unity. Then, the individual diagrams can be combined to one as shown in Fig. 2(b).

The power constraint of user *u* is denoted by $E_{\max}(u)$:

$$E_{\text{tot}}(u) = \sum_{n=1}^{N} E_{u,n} \le E_{\max}(u) \quad . \tag{8}$$

In the multiuser waterfilling diagram, the CNR curves of all users are combined to one curve by choosing the minimum. The equivalent energies are thus given by

$$\hat{E}_{u,n} = \begin{cases} [1 - b_u T_{u,n}^{-1}]^{+} & \text{for } b_u T_{u,n}^{-1} \le b_l T_{l,n}^{-1} \quad \forall l \neq u \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

with

$$\sum_{n=1}^{N} \hat{E}_{u,n} \le b_u \cdot E_{max}(u) \,,\ \forall u = 1, \ldots, U \quad . \tag{10}$$

Inserting (9) into (10) gives a system with *U* equations for the *U* multipliers $b_1, \ldots, b_U$. If this system is solved for the b_u, the energy allocation is given by (9) and the subchannel allocation can be derived easily. Unfortunately, the equation system is highly nonlinear and standard algorithms, which have been applied to this system, have not produced satisfying results. In the next section we will derive an iterative algorithm which yields a good approximation of the solution.

Note, that for the case that a total power constraint is given instead of user-individual power constraints, the subcarrier allocation becomes a trivial task: for each subchannel the user with the highest CNR $T_{u,n}$ is chosen.

III. Multiuser Waterfilling Algorithm

A. Implementation

We observe from (9): the equivalent transmit power $\hat{E}_{\text{tot}}(u) = \sum_{n=1}^{N} \hat{E}_{u,n}$ increases for falling b_u. At the same time, the equivalent power budget $\hat{E}_{\max}(u) = b_u \cdot E_{\max}(u)$ decreases. For the values

$$b_u = \hat{b}_n = T_{u,n} \cdot \min_{l \neq u} \{ b_l T_{l,n}^{-1} \} \,,\ n = 1, \ldots, N \tag{11}$$

the equivalent transmit power $\hat{E}_{\text{tot}}(u)$ has a saltus, because an additional subcarrier is assigned to user *u* for falling b_u or a subcarrier is taken away for raising b_u. In the proposed algorithm, the multipliers b_u are varied until (10) is fulfilled for all users with (almost) equality. This approach can be regarded as a generalization of the two-user algorithm presented by Diggavi [7].

We define the subcarrier allocation matrix $\mathbf{A} = (a_{u,n})$ with $a_{u,n} = 1$ if user *u* is active on subcarrier *n*, and $a_{u,n} = 0$ otherwise. This matrix can be derived easily out of the values of $\mathbf{b} = (b_1, \ldots, b_U)$ and $\mathbf{T} = (T_{u,\nu})$:

$$\mathbf{A} = \boldsymbol{g}(\mathbf{b}, \mathbf{T}) = \begin{cases} 1 & \text{for } b_u T_{u,n}^{-1} \le b_l T_{l,n}^{-1} \quad \forall l \neq u \\ 0 & \text{otherwise} \end{cases} \quad . \tag{12}$$

At the beginning, the algorithm assigns arbitrary values to the multipliers $b_1, \ldots, b_U$. Then these values are varied for each user until

$$\Delta\hat{E} = \left(\sum_{n=1}^{N} \hat{E}_{u,n}\right) - b_u E_{\max}(u) \approx 0 \quad . \tag{13}$$

For $\Delta\hat{E} > 0$ the energy assigned to user *u* is too big and consequently b_u must be increased and a subcarrier is taken away from user *u*. For $\Delta\hat{E} < 0$, b_u must be lowered and eventually an additional subcarrier is assigned to user *u*. In order to accelerate this process, the saltuses $\hat{b}_n$ are calculated according to (11) and depending on the sign of $\Delta\hat{E}$, the next bigger or the next smaller value out of the set $\{\hat{b}_1, \ldots, \hat{b}_N\}$ is assigned to b_u. It may happen that there is no smaller value for

$\mathbf{b} = \mathbf{1}$, $\mathbf{A} = \boldsymbol{g}(\mathbf{b}, \mathbf{T})$
repeat
  $\mathbf{A}_{old} = \mathbf{A}$
  for $u = 1,\ldots,U$
    $\Delta\hat{E} = \sum_{n=1}^{N} a_{u,n}[1 - b_u T_{u,n}^{-1}]^{+} - b_u E_{\max}(u)$
    if $\Delta\hat{E} \neq 0$ then
      $\hat{b}_n = T_{u,n} \cdot \min_{l \neq u}\{b_l T_{l,n}^{-1}\}$, $\forall n = 1, \ldots, N$
      repeat
        $b_{\text{old}} = b_u$
        if $\Delta\hat{E} > 0$ then
          $b_u = \min\{\hat{b}_n | \hat{b}_n > b_{\text{old}}\}$ // next bigger $\hat{b}_n$
          $n_1 = \arg\min\{\hat{b}_n | \hat{b}_n > b_{\text{old}}\}$
          $a_{u,n_1} = 0$
        else if
          $b_u = \max\{\hat{b}_n | \hat{b}_n < b_{\text{old}}\}$ // next smaller $\hat{b}_n$
          if $\max\{\hat{b}_n | \hat{b}_n < b_{\text{old}}\} = \{\}$ then
            $b_u = (1 - \varepsilon) b_{\text{old}}$
          else if
            $n_1 = \arg\max\{\hat{b}_n | \hat{b}_n < b_{\text{old}}\}$
            $a_{u,n_1} = 1$
          end if
        end if
        $\Delta\hat{E}_{\text{old}} = \Delta\hat{E}$
        $\Delta\hat{E} = \sum_{n=1}^{N} a_{u,n}[1 - b_u T_{u,n}^{-1}]^{+} - b_u E_{\max}(u)$
      until $\operatorname{sgn}(\Delta\hat{E}_{\text{old}}) \neq \operatorname{sgn}(\Delta\hat{E})$
      $b_u = (b_{\text{old}}\Delta\hat{E} - b_u \Delta\hat{E}_{\text{old}})/(\Delta\hat{E} - \Delta\hat{E}_{\text{old}})$
      $\mathbf{A} = \boldsymbol{g}(\mathbf{b}, \mathbf{T})$
    end if
  end
until $\mathbf{A} = \mathbf{A}_{\text{old}}$

Fig. 3. Algorithm for multiuser waterfilling.

b_u in the set. In this case, the multiplier b_u is decreased by a small amount without assigning a new subcarrier. If $\Delta\hat{E} > 0$, the value of b_u is increased stepwise until a sign change in $\Delta\hat{E}$ is encountered. Note that this implicitly fulfills the condition (13). For $\Delta\hat{E} < 0$, the same applies accordingly. The optimal value for b_u lies between the two last values and is approximated linearly with

$$b_u^{(\text{new})} = \frac{b_{\text{old}}\Delta\hat{E} - b_u\Delta\hat{E}_{\text{old}}}{\Delta\hat{E} - \Delta\hat{E}_{\text{old}}} . \tag{14}$$

Then, with the new values of $\boldsymbol{b}$ the subcarrier allocation matrix **A** is determined and the algorithm continues with the next user. This procedure is repeated until the allocation matrix is stable.

B. Channel Capacity

The multiuser waterfilling algorithm determines the multipliers **b** and the subcarrier allocation matrix **A**. Thus, the symbol energies can be calculated as

$$E_{u,n} = \frac{a_{u,n}}{b_u} \cdot [1 - b_u T_{u,n}^{-1}]^+ \tag{15}$$

and the signal power at the receiver is

$$E_{\text{rx}}(n) = \sum_{u=1}^{U} E_{u,n} |H_{u,n}|^2 . \tag{16}$$

Thus, the channel capacity for the multiple access channel is

$$C = \sum_{n=1}^{N} \log_2\left(1 + \frac{E_{\text{rx}}(n)}{\sigma_n^2}\right) . \tag{17}$$

IV. Simulation Results

The channel transfer functions for four users were generated assuming a wireless channel as described in [8] and are depicted in Fig. 4. White noise was assumed, i.e. $\sigma_n^2 = N_0 \forall n$; thus the CNRs are just the inverse of the squared magnitude response. For these CNRs and a power budget[1] of 15 dB per user the proposed algorithm was executed. This algorithm produces the subcarrier allocation matrix **A** which is subsequently input to a single-user bitloading algorithm which assigns the appropriate bit and power to each subchannel user by user. For this task, an algorithm which approximates the single-user waterfilling theorem [9] was applied, which yields the bit allocation as shown in Fig. 5. Note that this already fixes the transmit power through (1) and (7) as

$$E_{u,n} = T_{u,n}^{-1}(2^{b_{u,n}} - 1) . \tag{18}$$

1. All powers are normalized to the total noise power $P_{\text{N}} = \sum_{n=1}^{N} \sigma_n^2$

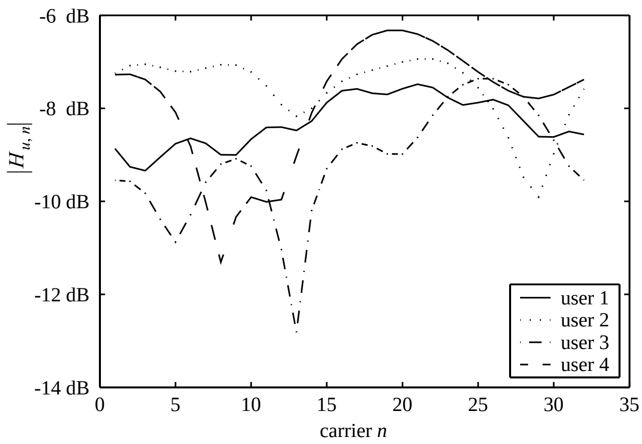

Fig. 4. Transfer functions for four users.

The operation of the algorithm can be visualized easily by using the multipliers b_u which are produced as a by-product. The equivalent symbol energy per subchannel at the transmitter side is given by

$$\hat{E}_{\text{tx}}(n) = \sum_{u=1}^{U} a_{u,n} \cdot [1 - b_u T_{u,n}^{-1}]^+ . \tag{19}$$

These values are added in the waterfilling diagram in Fig. 6 to the minimum equivalent inverse CNR $b_u T_{u,n}^{-1}$ and the sum represents the water level which adds up to one for all subcarriers.

The SNR gap for this simulation was chosen $\Gamma_u = 2$, which corresponds to a SER of $P_{\text{S}} = 2 \cdot 10^{-4}$ and a coding gain of 4 dB.

The channel capacity of this multiple access channel was evaluated according to (17) as $C = 161$ bit per OFDM symbol and the total achieved bitrate (which corresponds to the sum of

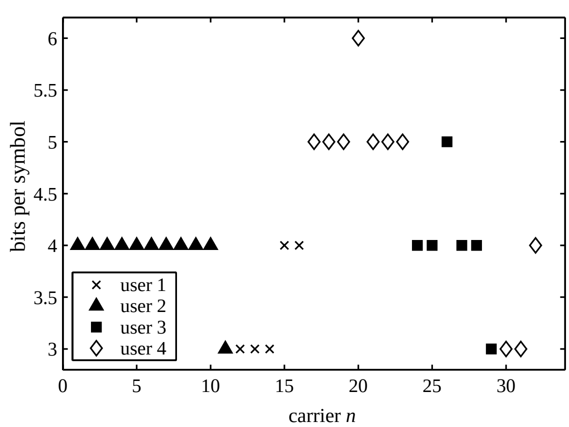

Fig. 5. Resulting bit allocation after subcarrier and bit allocation.

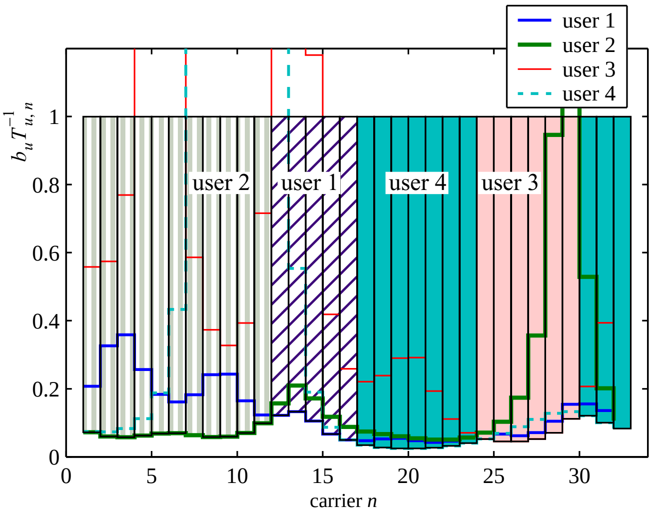

Fig. 6. Waterfilling diagram for $N = 32$ subcarriers and $U = 4$ users. The bottom line of the diagram is given by $\min_u\{b_u T_{u,n}^{-1}\}$. Onto this line the symbol energy $\hat{E}_{\text{tx}}(n)$ according to (19) is added.

the bits per subchannel shown in Fig. 5) was 130 bit per OFDM symbol. The discrepancy between channel capacity and bitrate depends heavily on the SNR gap – only for the impractical value $\Gamma_u = 1$ the difference approaches zero.

The presented algorithm was tested with a great number of different CNRs, different numbers of users and subchannels. The low number of subcarriers and users in the example shown in Fig. 6 was chosen for illustrative purposes. The computational complexity depends approximately linearly on the number of subcarriers, making this algorithm well suited for multicarrier systems with a high number of carriers. The algorithm presented in [4] is reported to have a complexity which is proportional to $N \log N$.

V. Conclusion

We have presented in detail an efficient algorithm for subcarrier allocation in multiple access OFDM systems. The proposed algorithm is based on the multiuser waterfilling theorem of information theory, and it maximizes the total bitrate under the constraint of a maximum transmit power for each user. We have described how this subcarrier allocation algorithm can be combined with existing solutions for single-user bitloading and have presented simulation results for wireless multiple access channels. A great advantage of the proposed algorithm is its low complexity which is proportional to N compared to $N \log N$ of existing solutions, where N is the number of OFDM subcarriers.

References

[1] R. G. Gallager, *Information Theory and Reliable Communications*. New York: Wiley, 1968.

[2] R. V. Sonalkar, "An efficient bit-loading algorithm for DMT applications," *IEEE Comm. Letters*, vol. 4, no. 3, pp. 80-82, March 2000.

[3] R. S. Cheng, S. Verdú, "Gaussian multiaccess channels with ISI: capacity region and multiuser water-filling," *IEEE Trans. Information Technology*, vol. 39, no. 3, May 1993.

[4] C. Zeng, L. M. C. Hoo, J. M. Cioffi, "Efficient water-filling algorithms for a Gaussian multiaccess channel with ISI," *VTC Fall 2000*, Boston, Sept. 2000.

[5] C. Y. Wong, R. S. Cheng, K. B. Letaief, R. D. Murch, "Multiuser OFDM with adaptive subcarrier, bit, and power allocation," *IEEE JSAC*, vol. 17, no. 10, pp. 1747-1757, Oct. 1999.

[6] J. M. Cioffi, "A multicarrier primer," *ANSI T1E1.4 Committee Contribution*, Nov. 1991.

[7] S. N. Diggavi, "Multiuser DMT: a multiple access modulation scheme," *Globecom '96*, London, Nov. 1996.

[8] P. Höher, "A statistical discrete-time model for the WSSUS multipath channel," *IEEE Trans. Vehicular Tech.*, vol. 41, no. 4, Nov. 1992.

[9] D. Hughes-Hartoges, "Ensemble modem structure for imperfect transmission media," *U.S. Patent 4,679,227*, July 1987.