Analysis of System Performance

IN2072

Chapter 3 – Markov Chains

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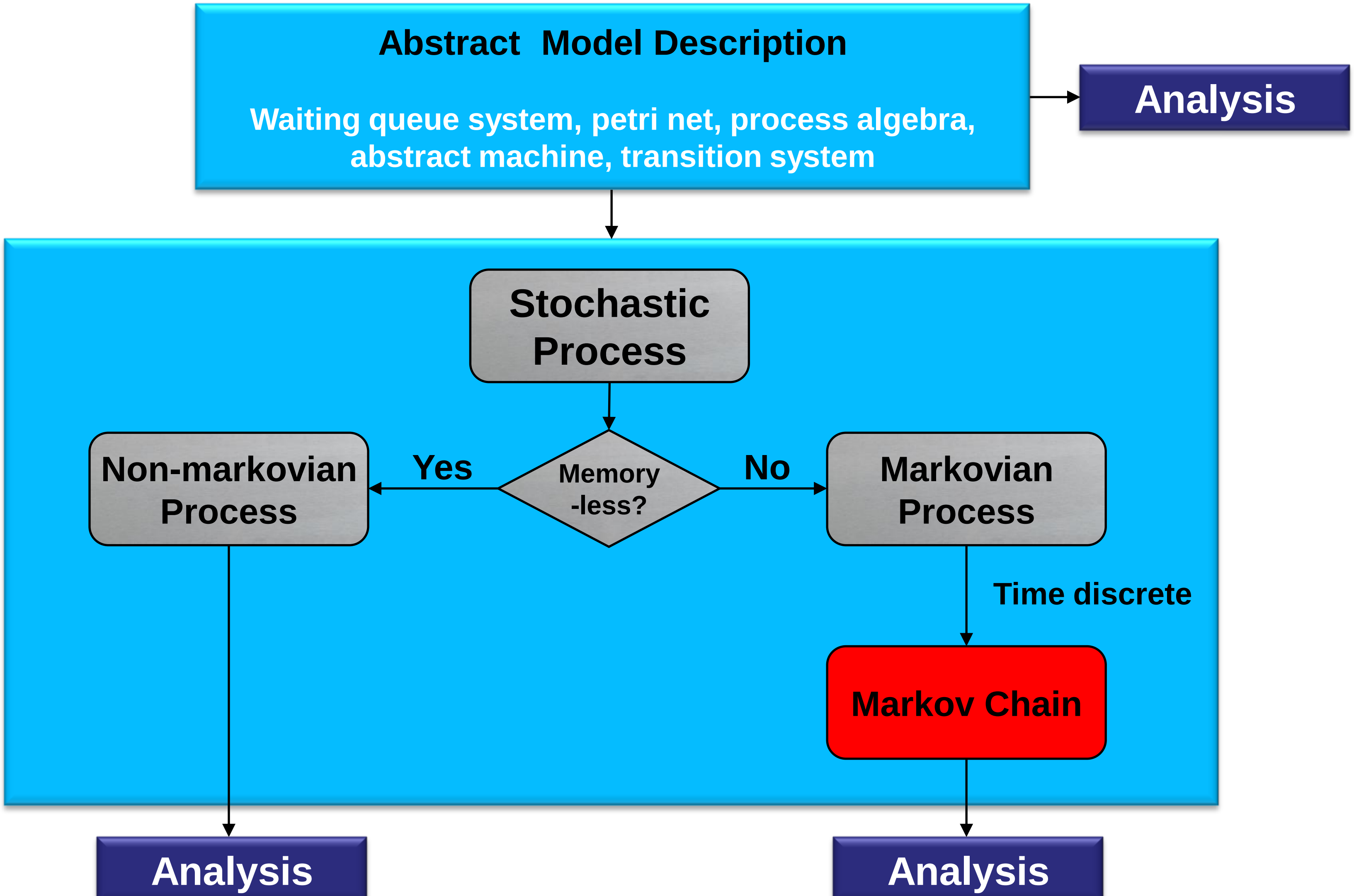
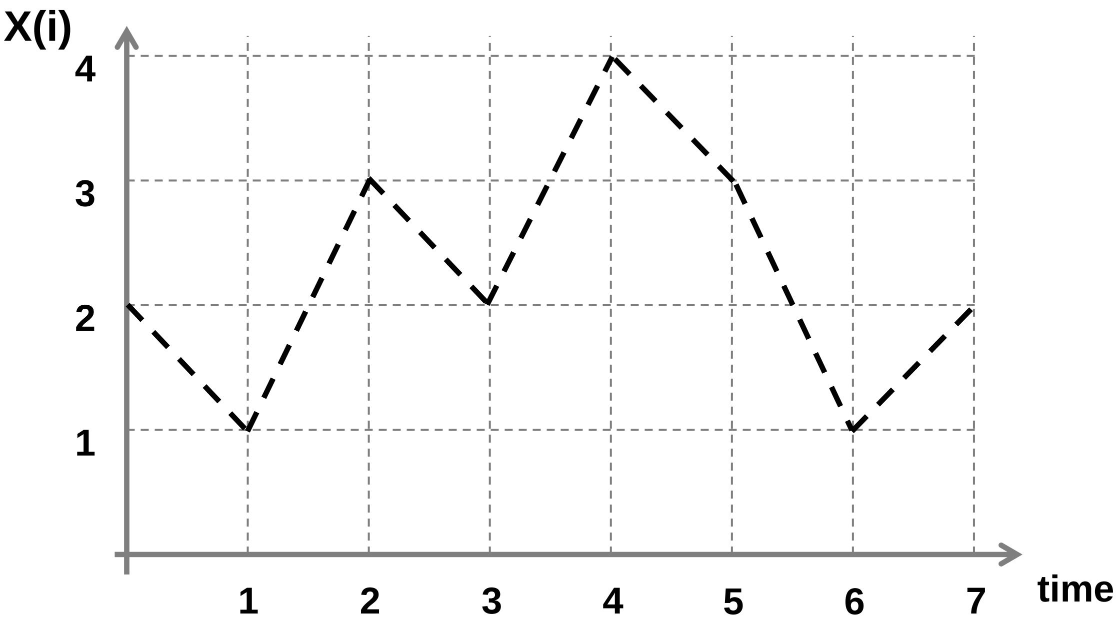
Classification



Stochastic process

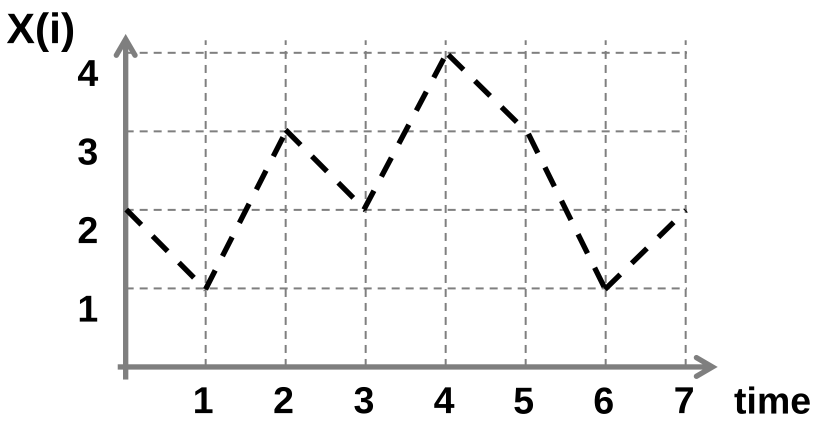
- Process development

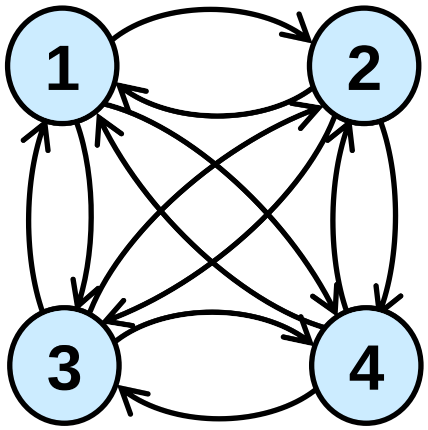


Process trajectory is given by the following expression:

$$\Rightarrow \quad P\left\{X(t_{n+1}) = x_{n+1} \middle| X(t_n) = x_n, \ldots, X(t_0) = x_0\right\}$$

- Process development





Process trajectory is given by the following expression:

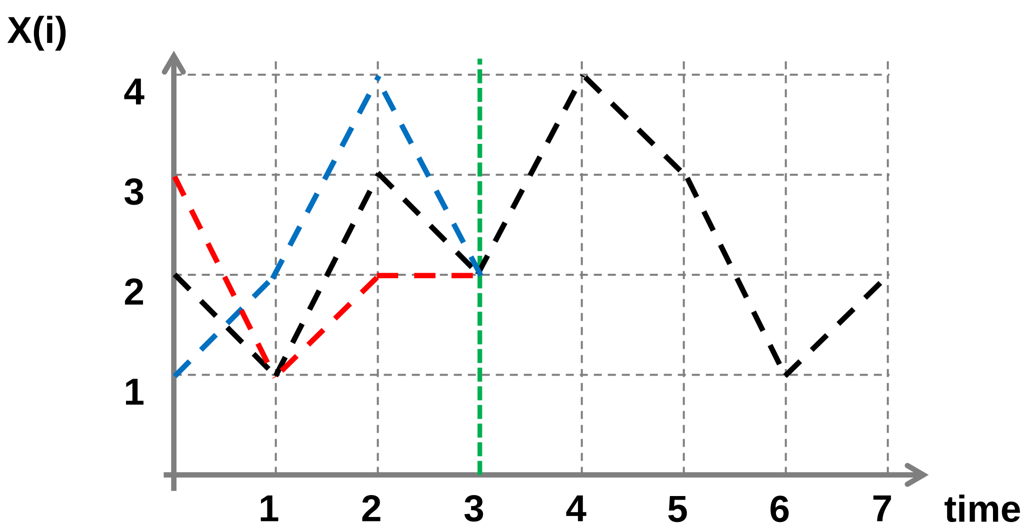
$$\Rightarrow \quad P\left\{X(t_{n+1}) = x_{n+1} \middle| X(t_n) = x_n, \ldots, X(t_0) = x_0\right\}$$

Markovian process

Transient behavior of markovian processes:

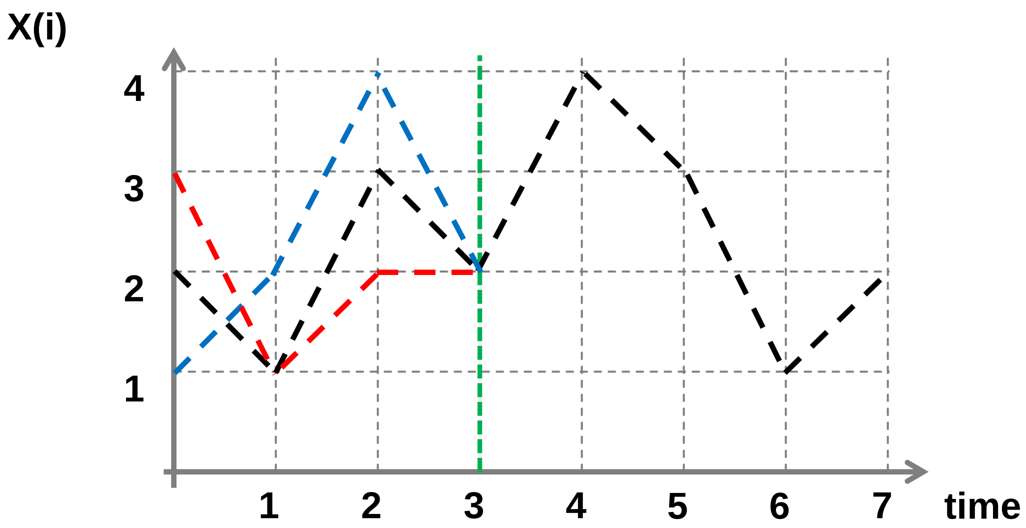
The future development of a markovian process **only** depends on its current state and not on its behavior in the past.

$$P\left\{X(t_{n+1}) = x_{n+1} \middle| X(t_n) = x_n, \ldots, X(t_0) = x_0\right\} =$$
$$P\left\{X(t_{n+1}) = x_{n+1} \middle| X(t_n) = x_n\right\}, t_0 < t_1 < \ldots < t_n < t_{n+1}.$$



- Markov chain:

  A markov chain is a markovian process with finite or countable (discrete) state space.



Discrete Time Markov Chain (DTMC)

- A DTMC evolves over time, that is, step by step, according to one-step transition probabilities.

- Transition probability:

  The probability that the process changes from state i to state j within a single process step is given by:

Superscript corresponds to the number of process ticks

$$\Rightarrow \quad p_{ij}^{(1)}(n) = P\{X(t_{n+1}) = x_{n+1} = j \mid X(t_n) = x_n = i\}$$

- (One-step) Transition probability matrix:

$$\Rightarrow \quad P = P^{(1)} = [p_{ij}] = \begin{pmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ p_{20} & p_{21} & p_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with $\sum_j p_{ij} = 1$

and $0 \leq p_{ij} \leq 1$

Classification of DTMCs

- Definition:

  Any state j is said to be reachable from any other state i, where $i, j \in S$, if it is possible to transit from state i to state j in a finite number of steps according to the given transition probability matrix. For some integer $n \geq 1$, the following relation must hold for the n-step transition probability:

  $$\Rightarrow \quad p_{ij}^{(n)} > 0, \qquad \exists n, n \geq 1$$

- Irreducible:

  A DTMC is called irreducuble if all states in the chain can be reached pairwise from each other.

  $$\Rightarrow \quad \forall i, j \in S, \qquad \exists n, n \geq 1 : p_{ij}^{(n)} > 0$$

- Absorbing:

  A state is called absorbing state if and only if no other state of the DTMC can be reached from it. $\Rightarrow$ $p_{ii} = 1$

Discrete Time Markov Chain (DTMC)

Example:

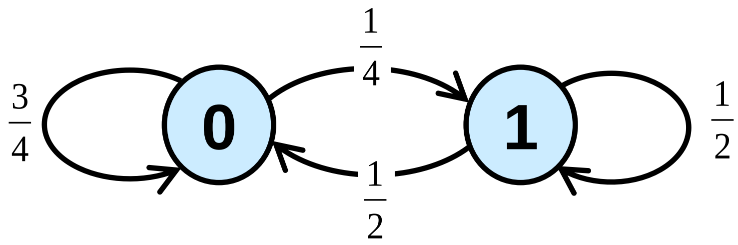
Consider a system with two states, e.g. a CPU which can be either idle or busy.

- The state space of the system is modelled as $S = \{0,1\}$.

- The one-step transition probability matrix of this two-state DTMC is given by:

$$\Rightarrow \quad P^{(1)} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{pmatrix}$$

- Its behavior can be represented by the following finite directed graph:



Discrete Time Markov Chain (DTMC)

- **N-step transition probability:**

  Is the probability that markov chain transits from state i at time k to state j at time l in exactly $n = l - k$ steps.

$$\Rightarrow \quad p_{ij}^{(n)}(k,l) = P\{X(t_l) = x_l = j \mid X(t_k) = x_k = i\}, \qquad 0 \le k \le l$$

with $\displaystyle\sum_j p_{ij}^{(n)}(k,l) = 1$ and $0 \le p_{ij}^{(n)}(k,l) \le 1$

Discrete Time Markov Chain (DTMC)

- Idea:

  Compute the n-step transition probabilities recursively from the one-step transition probabilities.

  ⇨ Split the transition from state i at time k to state j at time l into sub-transitions from state i at time k to a state h at time m and from state h at time m to state j at time l, where k<m<l and n=l-k.

$$p_{ij}^{(n)}(k,l) = \sum_{h \in S} p_{ih}^{(m-k)}(k,m) \cdot p_{hj}^{(l-m)}, \qquad 0 \le k \le l$$

Discrete Time Markov Chain (DTMC)

- Homogeneous DTMC:

  Behaviour of DTMC is not time-dependent.

$$p_{ij}^{(n)} = p_{ij}^{(n)}(k,l)$$

$p_{ij}^{(n)}$ only depends on the difference $n = l - k$ and not on the absolute values of k and l.

$$p_{ij}^{(n)} = P\{X_{k+n} = j \mid X_k = i\} = P\{X_n = j \mid X_0 = i\}, \qquad \forall k \in T$$

$$p_{ij}^{(n)} = \sum_{h \in S} p_{ih}^{(m)} \cdot p_{hj}^{(n-m)}, \qquad 0 \le m \le n$$ Chapman-Kolmogorov

$$p_{ij}^{(n)} = \sum_{h \in S} p_{ih}^{(1)} \cdot p_{hj}^{(n-1)}, \qquad m \le n$$

Start state and number of time steps are sufficient for the calculation.

Discrete Time Markov Chain (DTMC)

- Homogeneous DTMC:

$$\Rightarrow \quad p_{ij}^{(n)} = \sum_{h \in S} p_{ih}^{(1)} \cdot p_{hj}^{(n-1)}, \qquad m \le n$$

Start state and number of time steps are sufficient for the calculation.

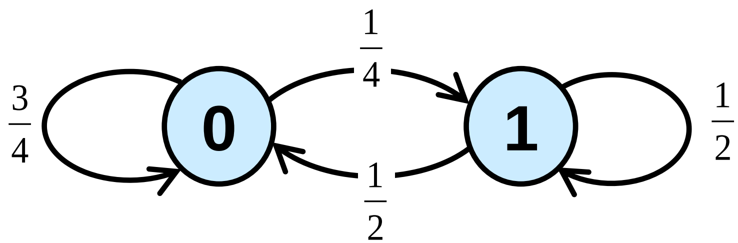
With $P^{(n)}$ as the matrix of n-step transition propabilities $p_{ij}^{(n)}$, we can formulate the Chapman-Kolmogorov equation from the previous slide as:

$$\Rightarrow \quad P^{(n)} = P^{(1)} \cdot P^{(n-1)} = P \cdot P^{(n-1)} = P^n$$

The n-step transition propability matrix can be computed by the (n-1)-fold multiplication of the one-step transition matrix by itself.

Discrete Time Markov Chain (DTMC)

- Example:



One step transition probability matrix:

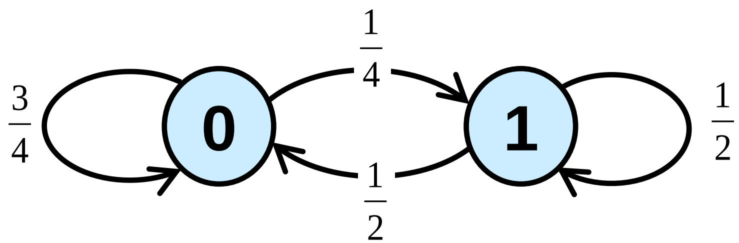
$$\Rightarrow \quad P^{(1)} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{pmatrix}$$

Four step transition propability matrix:

$$\Rightarrow \quad P^{(4)} = \begin{pmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{pmatrix}^4$$

Discrete Time Markov Chain (DTMC)

- Example:



Four step transition propability matrix:

$$\Rightarrow \quad P^{(4)} = P \cdot P^{(3)} = P^{2} \cdot P^{(2)} \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{pmatrix}^{2} \cdot P^{(2)}$$

$$= \begin{pmatrix} 0.6875 & 0.3125 \\ 0.625 & 0.375 \end{pmatrix} P \cdot P^{(1)} = \begin{pmatrix} 0.67188 & 0.32813 \\ 0.65625 & 0.34375 \end{pmatrix} P^{(1)}$$

$$= \begin{pmatrix} 0.66797 & 0.33203 \\ 0.66406 & 0.33594 \end{pmatrix}$$

Discrete Time Markov Chain (DTMC)

- Goal:

  Compute the probability mass function of the random variable X_n, that is, the probabilities $v_i(n) = P\{X_n = i\}$ that the DTMC is in state i at time step n.

  Vector of state probabilities at time n

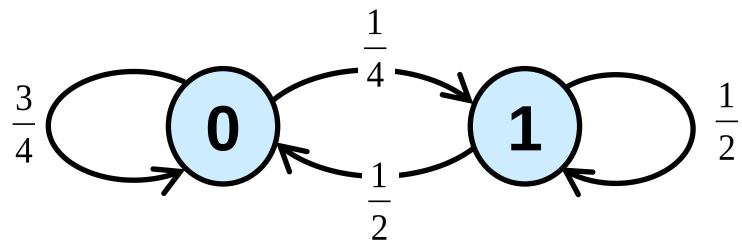
$$v(n) = \{v_0(n), v_1(n), v_2(n), \ldots\}$$

  can be obtained by un-conditioning the transition probability matrix $P^{(n)}$ on the initial probability vector $v(0) = \{v_0(0), v_1(0), v_2(0), \ldots\}$:

$$\Rightarrow \quad v(n) = v(0)P^{(n)} = v(0) \cdot P^n = v(n-1) \cdot P$$

Discrete Time Markov Chain (DTMC)

- Example:



We assume that the system is in state one which results in the initial probability vector $v^{(1)}(0) = (0,1)$.

$$\Rightarrow \quad v^{(1)}(4) = (0,1) \cdot \begin{pmatrix} 0.66797 & 0.33203 \\ 0.66406 & 0.33594 \end{pmatrix} = (0.66406, 0.33594)$$

- Example:

$$v^{(2)}(0) = \left(\frac{2}{3}, \frac{1}{3} \right) = (0.6\overline{6}, 0.3\overline{3})$$

$$\Rightarrow \quad v^{(1)}(4) = (0.6\overline{6}, 0.3\overline{3}) \cdot \begin{pmatrix} 0.66797 & 0.33203 \\ 0.66406 & 0.33594 \end{pmatrix} = (0.6\overline{6}, 0.3\overline{3})$$

Discrete Time Markov Chain (DTMC)

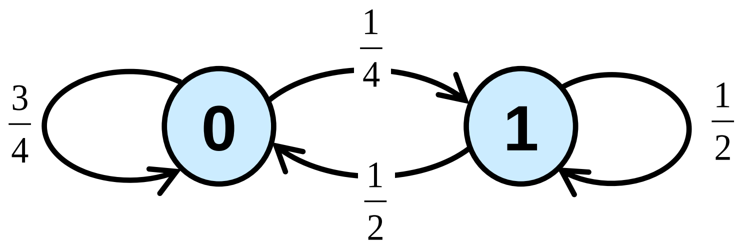
- Stationary state probabilities:

  State probability $\nu = (\nu_0, \nu_1, \ldots, \nu_i, \ldots)$ of a discrete-time Markov chain are said to be stationary, if any transitions of the underlying DTMC according to the given one-step transition propabilities $P = [p_{ij}]$ have no effect on these state probabilities, that is, $\nu_j = \sum_{i \in S} \nu_i p_{ij}$ holds all states $j \in S$. This relation can also be expressed in matrix form:

$$\Rightarrow \quad \nu = \nu P, \qquad \sum_{i \in S} \nu_i = 1$$

Discrete Time Markov Chain (DTMC)

- Example:



The n-step transition probabilities converge as n→∞ .

$$\Rightarrow \quad \tilde{P} = \lim_{n\to\infty} P^{(n)} = \lim_{n\to\infty} \begin{pmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{pmatrix}^n = \begin{pmatrix} 0.6\overline{6} & 0.3\overline{3} \\ 0.6\overline{6} & 0.3\overline{3} \end{pmatrix}$$

$$\Rightarrow \quad \tilde{v} = (0.6\overline{6}, 0.3\overline{3})$$

Discrete Time Markov Chain (DTMC)

- Transient analysis
  - Short-term behavior
  - State probabilities are time dependent
  - Initial state probability vector influences the state probabilities

- Steady-state analysis
  - Long-term behavior
  - State probabilities are time independent
  - Initial state probability vector does not affect the steady-state probabilities

> Transient analysis has special relevance if short-term behavior is of more importance than long-term behavior

DTMC Examples

- Example:

Transition probability matrix: $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

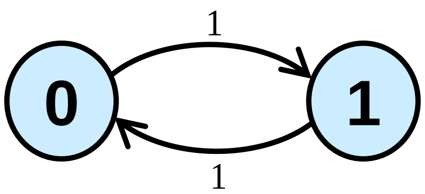
$$\Rightarrow \lim_{n\to\infty} P^{(n)} = \tilde{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \tilde{v} = v(0) \cdot \tilde{P} = v(0)$$

$\Rightarrow$ Limiting state probabilities $\tilde{v}$ do exist and are identical with the initial state probability vector $v(0)$.

DTMC Examples

- Example:



Transition probability matrix: $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

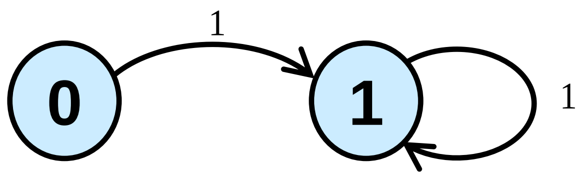
⇒ $\lim_{n\to\infty} P^{(n)} = \tilde{P}$ The n-step transition matrix $P^{(n)}$ does not converge.

⇒ Stationary probability vector $v = (0.5, 0.5)$

⇒ Limiting state probabilities $\tilde{v}$ do NOT exist since the n-step transition matrix $P^{(n)}$ does not converge.

DTMC Examples

- Example:



Transition probability matrix: $P = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow$ $\lim_{n\to\infty} P^{(n)} = \tilde{P}$ The n-step transition matrix $P^{(n)}$ converges.
(all TPMs are identical)

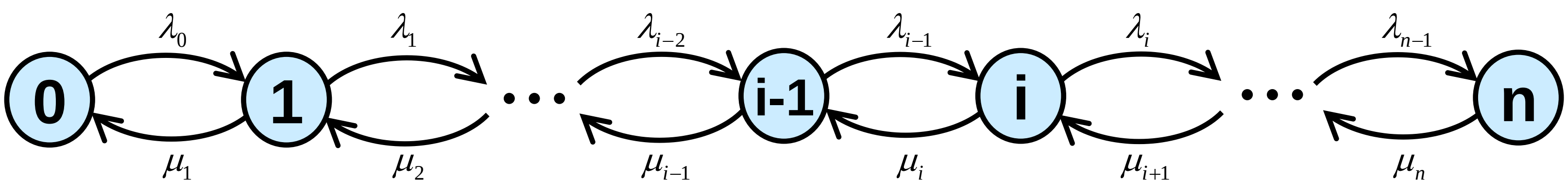
$\Rightarrow$ Stationary probability vector $v = (0,1)$

Birth-death process

- **Definition:**

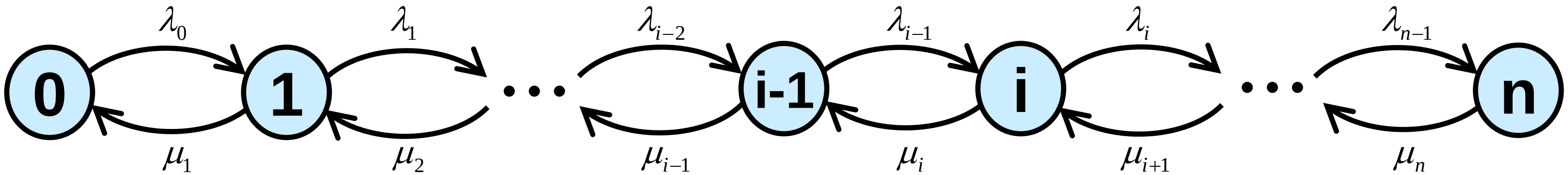
  Birth-death processes are markovian processes which only have transitions between two neighbor states.

  - Birth-death processes have usually one dimensional state spaces.
  - A markov model with multi-dimensional state space is also often referred to as birth-death process if there only exist transitions among neighbor states in each direction of the state space.



Birth-death process with finite state space

Birth-death process



Transition probability densities:

$$q_{ij} = \begin{cases} \lambda_i & i = 0,1,\ldots,n-1 \quad j = i+1 \\ \mu_i & i = 1,2,\ldots,n \quad j = i-1 \\ 0 & else \end{cases}$$

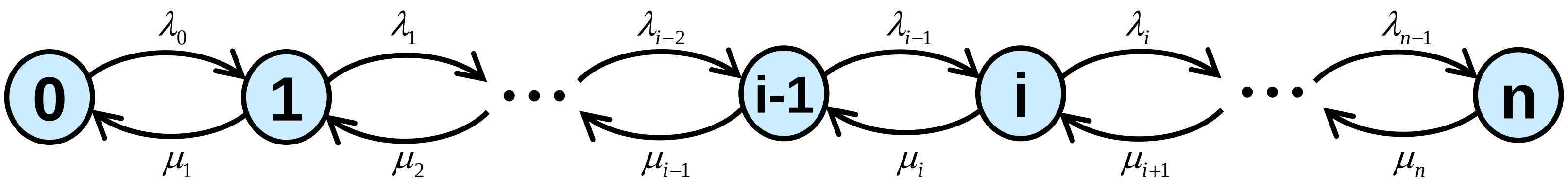
Special case:

$\mu_i = 0, \quad \forall i \in S$ Pure birth process

$$x(n) = P\{X = n\} = 1, \qquad x(i) = 0 \qquad else$$

$\lambda_i = 0, \quad \forall i \in S$ Pure death process

$$x(0) = P\{X = 0\} = 1, \qquad x(i) = 0 \qquad else$$



- Time-dependent probabilities of the birth-death process

$$\Rightarrow \quad \frac{\partial}{\partial t} x(0,t) = -\lambda_0 x(0,t) + \mu_1 x(1,t)$$

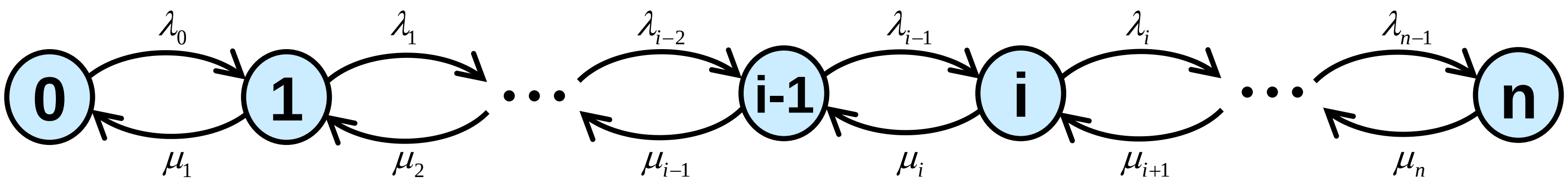
$$\Rightarrow \quad \frac{\partial}{\partial t} x(i,t) = -(\lambda_i + \mu_i) x(i,t) + \lambda_{i-1} x(i-1,t) + \mu_{i+1} x(i+1,t),$$

$$i = 1, \ldots, n-1$$

$$\Rightarrow \quad \frac{\partial}{\partial t} x(n,t) = -\mu_n x(n,t) + \lambda_{n-1} x(n-1,t)$$

Solving the differential system of equations with starting conditions $\{x(i,0), \quad i = 0, \ldots, n\}$ leads to the state probability vector at observation time t $\{x(i,t), \quad i = 0, \ldots, n\}$.

Stationary birth-death process



Equilibrium state of the system of equations of the **micro states** is given by:

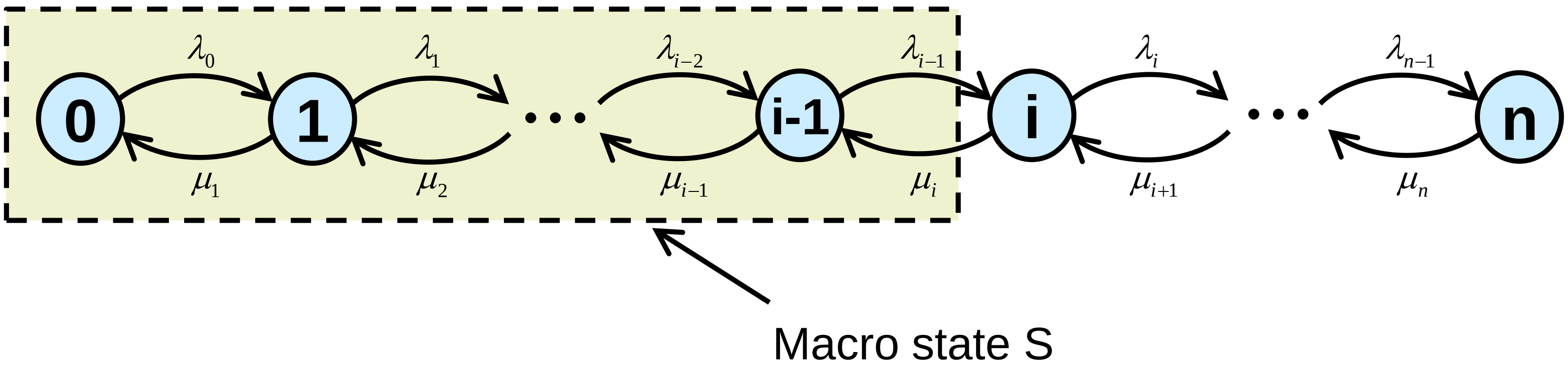
$$\lambda_0 x(0) = \mu_1 x(1)$$

$$(\lambda_i + \mu_i) x(i) = \lambda_{i-1} x(i-1) + \mu_{i+1} x(i+1), \qquad i = 1, 2, \ldots, n-1$$

$$\lambda_{n-1} x(n-1) = \mu_n x(n)$$

$$\sum_{i=0}^{n} x(i) = 1$$

Stationary birth-death process

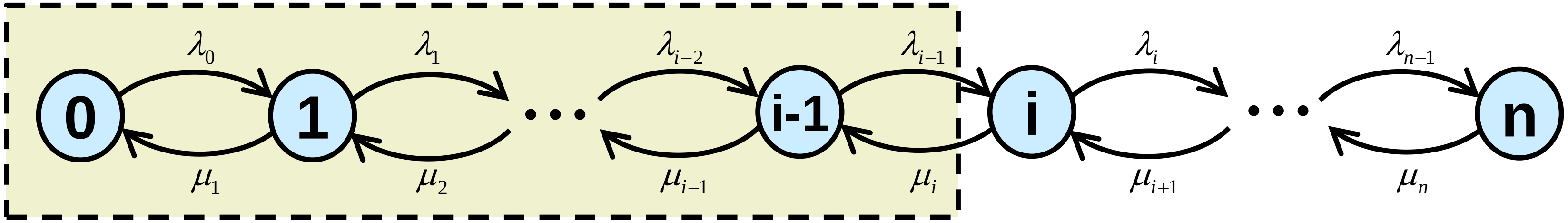


Macro state S consists of micro states $\{X = 0,1,\ldots,i-1\}$.

$$\Rightarrow \quad \lambda_{i-1}x(i-1) = \mu_i x(i), \qquad i = 1,2,\ldots,n$$

$$\Rightarrow \quad \sum_{i=0}^{n} x(i) = 1$$

Stationary birth-death process



This system of equations can be resolved by succesive insertion of the micro states.

$$\Rightarrow \quad x(i) = x(0) \cdot \frac{\prod_{k=0}^{i-1} \lambda_k}{\prod_{k=0}^{i} \mu_k}, \qquad i = 1,2,\ldots,n$$

The unknown state probability $x(0)$ can be calculated by using the normalization condition (total probability).

$$\Rightarrow \quad 1 = \sum_{i=0}^{n} x(i) = x(0) + x(0) \sum_{i=1}^{n} \frac{\prod_{k=0}^{i-1} \lambda_k}{\prod_{k=1}^{i} \mu_k} \qquad \Rightarrow \quad x(0)^{-1} = 1 + \sum_{i=1}^{n} \frac{\prod_{k=0}^{i-1} \lambda_k}{\prod_{k=1}^{i} \mu_k}$$

Questions

- What are the characteristics of a markov process?
- Differences between transient analysis and steady-state analysis.
- What is a chain?
- What is a markov chain?
- When does a system become stationary?
- What is an absorbing state?
- What is birth-death process?