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Tutorials for Network Coding (IN3300)
Tutorial 2 – 2014/10/30

Problem 1 Maximum flow problem

We consider the wired network with $n = 6$ nodes and $m = 7$ arcs that is described by the incidence matrix

$$\boldsymbol{M} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}.$$

Arcs are enumerated in lexicographic order as known from the lecture, e.g. $(1,2) < (2,1)$.

a)* Draw the network described by $\boldsymbol{M}$ and label both nodes and arcs.

b)* What is the rank $\boldsymbol{M}$?

c)* Determine a basis B of null $\boldsymbol{M}$.

The arc capacities are given by $\boldsymbol{z} = [2\ 2\ 2\ 2\ 1\ 2\ 2]$. We consider a single unicast between nodes 1 and 6 described by $\boldsymbol{d} = [1\ 0\ 0\ 0\ 0\ \ -1]$.

d)* Determine the capacity between nodes 1 and 6 using the min-cut / max-flow theorem. (A bit tedious to enumerate all the cuts …)

The maximum flow problem is formally expressed as linear program

$$\max_{r}\ r \quad \text{s.t.} \quad \boldsymbol{M}\boldsymbol{x} = r\boldsymbol{d}, \tag{1}$$
$$\boldsymbol{x} \geq \boldsymbol{0}, \tag{2}$$
$$\boldsymbol{x} \leq \boldsymbol{z}. \tag{3}$$

The demand vector $\boldsymbol{d}$ is chosen such that its positive and negative components sum to 1 and -1, respectively.

In order to solve this problem using Matlab we have to rewrite it as

$$\min_{\boldsymbol{x}} \ \boldsymbol{f}^T\boldsymbol{x} \quad \text{s.t.} \quad \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{a}, \tag{4}$$

$$\boldsymbol{B}\boldsymbol{x} = \boldsymbol{b}, \tag{5}$$

$$\boldsymbol{x} \ge 0, \tag{6}$$

$$\boldsymbol{x} \le \boldsymbol{z}. \tag{7}$$

e)* Express the scalar rate r by means of $\boldsymbol{M}$, $\boldsymbol{x}$, and $\boldsymbol{d}$.

f) Determine $\boldsymbol{B}$ such that $\boldsymbol{B}\boldsymbol{x} = \boldsymbol{0}$ is equivalent to $\boldsymbol{M}\boldsymbol{x} = r\boldsymbol{d}$.

g) Determine $\boldsymbol{f}$ such that $\boldsymbol{f}^T\boldsymbol{x} = r$.

h) State the revised optimization problem and solve it using Matlab.

Now we consider the case of a second flow, i. e., we have two demand vectors

$$\boldsymbol{d}_1 = [1\ 0\ 0\ 0\ 0\ \ -1]^T \text{ and}$$
$$\boldsymbol{d}_2 = [0\ 1\ 0\ 0\ \ -1\ 0]^T.$$

If we want to maximize the joint rate $r = r_1 + r_2$, the optimization problem becomes

$$\max_{r_1, r_2} \ r_1 + r_2 \quad \text{s.t.} \quad \boldsymbol{M}\boldsymbol{x}_1 = r_1\boldsymbol{d}_1, \tag{8}$$

$$\boldsymbol{M}\boldsymbol{x}_2 = r_2\boldsymbol{d}_2, \tag{9}$$

$$\boldsymbol{x}_1, \boldsymbol{x}_2 \ge \boldsymbol{0}, \tag{10}$$

$$\boldsymbol{x}_1 + \boldsymbol{x}_2 \le \boldsymbol{z}. \tag{11}$$

We now restate this problem to solve it in Matlab. To this end, we define

$$\boldsymbol{N} = \begin{bmatrix} \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M} \end{bmatrix}, \ \boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix}, \ \boldsymbol{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \ \text{and} \ \boldsymbol{D} = \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{d}_2 \end{bmatrix}.$$

i) Express $\boldsymbol{r}$ by $\boldsymbol{N}$, $\boldsymbol{D}$, and $\boldsymbol{x}$.

j) Determine $\boldsymbol{B}$ such that $\boldsymbol{B}\boldsymbol{x} = \boldsymbol{0}$ is equivalent to $\boldsymbol{N}\boldsymbol{x} = \boldsymbol{D}\boldsymbol{r}$.

k) Determine $\boldsymbol{A}$ such that $\boldsymbol{A}\boldsymbol{x} \le \boldsymbol{z}$ describes the joint capacity constraint.

l) Determine $\boldsymbol{f}$ such that $\boldsymbol{f}^T\boldsymbol{x} = r_1 + r_2$.

m) State the final problem and solve it in Matlab.

n) Sketch the achievable rate region.