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Tutorials for Network Coding (IN3300)

Tutorial 3 – 2014/11/18

Problem 1 Lossy wireless networks

We consider the three-node wireless relay network $G = (N, H)$ depicted in Figure 1 in the lossy hypergraph model with orthogonal MAC.

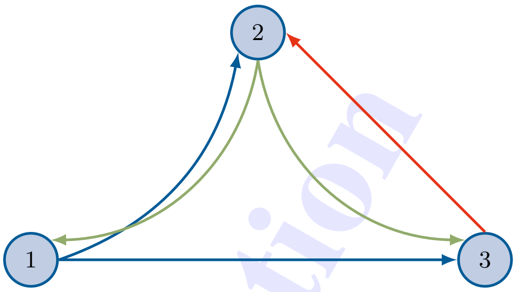


Figure 1: Three-node relay network

a)* Explicitly state the set of hyperarcs H.

N	H
1	$(1, \{2\}), (1, \{3\}), (1, \{2, 3\})$
2	$(2, \{1\}), (2, \{3\}), (2, \{1, 3\})$
3	$(3, \{2\})$

$$H = \{(1, \{2\}), (1, \{3\}), (1, \{2, 3\}), (2, \{1\}), (2, \{3\}), (2, \{1, 3\}), (3, \{1\})\}$$

b) Number the hyperarcs $(a, B) \in H$ in lexicographic ascending order, i.e., $(a, B) < (a', B')$ if

1. $a < a'$ *or*
2. $a = a' \ \wedge \ |B| < |B'|$ *or*
3. $a = a' \ \wedge \ |B| = |B'| \ \wedge \min B < \min B'$,

such that $j \equiv (a, B)$ with $j \in \{1, 2, \ldots\}$ for all $(a, B) \in H$.

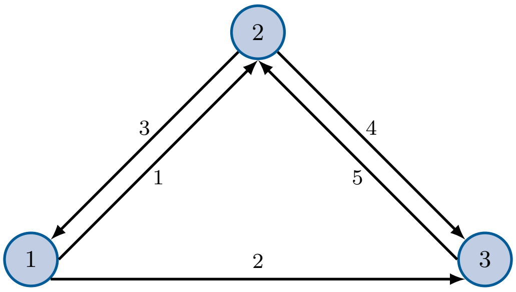
$(a, B) \in H$	$j \equiv (a, B)$	(a, b)	A_j
$(1, \{2\})$	1	$(1, 2)$	$\{1\}$
$(1, \{3\})$	2	$(1, 3)$	$\{2\}$
$(1, \{2, 3\})$	3	$(1, 2), (1, 3)$	$\{1, 2\}$
$(2, \{1\})$	4	$(2, 1)$	$\{3\}$
$(2, \{3\})$	5	$(2, 3)$	$\{4\}$
$(2, \{1, 3\})$	6	$(2, 1), (2, 3)$	$\{3, 4\}$
$(3, \{2\})$	7	$(3, 2)$	$\{5\}$

Note: The third column shows the solution for (c). The fourth column denotes the arc indices of $(a, b) \in A$ as determined in (e).

c)* Explicitly state all arcs $(a, b) \in A$ that are induced by each of the hyperarcs $(a, B) \in H$.

See solution of (b).

d) Draw the graph $G' = (N, A)$ that is induced by G.



(Numbers next to arcs denote the arc index $k \equiv (a, b) \in A$, which is done in (e).)

e) Number the arcs $(a, b) \in A$ in lexicographic ascending order, i.e., $(a, b) < (a', b')$ if

1. $a < a'$ *or*
2. $a = a' \wedge b < b'$,

such that $k \equiv (a, b)$ with $k \in \{1, 2, \dots\}$ for all $(a, b) \in A$. Also state by which hyperarc $j \equiv (a, B) \in H$ a given arc $k \equiv (a, b) \in A$ is induced by.

$(a, b) \in A$	$k \equiv (a, b)$
$(1, 2)$	1
$(1, 3)$	2
$(2, 1)$	3
$(2, 3)$	4
$(3, 2)$	5

f) Enumerate the sets A_j for all $j \equiv (a, B) \in H$ such that $(a, b) \equiv k \in A_j$ if hyperarc j induces arch k.

See solution of (c), fourth column.

g) State the hyperarc-arc incidence matrix $\boldsymbol{N}$.

$$\boldsymbol{N} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

h) State the incidence matrix $\boldsymbol{M}$ for G'.

$$\boldsymbol{M} = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 1 & -1 \\ 0 & -1 & 0 & -1 & 1 \end{bmatrix}$$

i) State the hyperarc-hyperarc incidence matrix $\boldsymbol{Q}$.

$$\boldsymbol{Q} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Assume that each arch $k \in A$ has unit capacity and a link error probability of $0 \leq \epsilon_k \leq 1$.

j) Determine the hyperarc capacity region $\mathcal{Z}$.

$$\mathcal{Z} = \bigcup_{\substack{\boldsymbol{\tau} \geq \mathbf{0} \\ \mathbf{1}^T \boldsymbol{\tau} \leq 1}} \left\{ \boldsymbol{z} : z_j = \tau_{\text{Tail}(j)} \prod_{l \in A_j} (1 - \epsilon_l) \prod_{\substack{l \notin A_j \\ \text{tail}(l) = \text{Tail}(j)}} \epsilon_l \quad \forall j \in H \right\}$$

$$\boldsymbol{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_7 \end{bmatrix} = \begin{bmatrix} \tau_1(1-\epsilon_1)\epsilon_2 \\ \tau_1(1-\epsilon_2)\epsilon_1 \\ \tau_1(1-\epsilon_1)(1-\epsilon_2) \\ \tau_2(1-\epsilon_3)\epsilon_4 \\ \tau_2(1-\epsilon_4)\epsilon_3 \\ \tau_2(1-\epsilon_3)(1-\epsilon_4) \\ \tau_3(1-\epsilon_5) \end{bmatrix}$$

k) Determine the broadcast capacity vector $\boldsymbol{y}$.

$$\boldsymbol{y} = \boldsymbol{Q}\boldsymbol{z} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{bmatrix} = \begin{bmatrix} z_1 + z_3 \\ z_2 + z_3 \\ z_1 + z_2 + z_3 \\ z_4 + z_6 \\ z_5 + z_6 \\ z_4 + z_5 + z_6 \\ z_7 \end{bmatrix} = \begin{bmatrix} \tau_1(1-\epsilon_1) \\ \tau_1(1-\epsilon_2) \\ \tau_1(1-\epsilon_1\epsilon_2) \\ \tau_2(1-\epsilon_3) \\ \tau_2(1-\epsilon_4) \\ \tau_2(1-\epsilon_3\epsilon_4) \\ \tau_3(1-\epsilon_5) \end{bmatrix}$$

l) Explicitly state the lossy hyperarc flow bound.

$$\boldsymbol{N}\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \\ x_3 \\ x_4 \\ x_3 + x_4 \\ x_5 \end{bmatrix} \leq \begin{bmatrix} z_1 + z_3 \\ z_2 + z_3 \\ z_1 + z_2 + z_3 \\ z_4 + z_5 \\ z_5 + z_6 \\ z_4 + z_5 + z_6 \\ z_7 \end{bmatrix}$$

m) Enumerate all $s - t$ cuts S and their respective capacities $v(S_i)$ for $s = 1$ and $t = 3$.

$$
\begin{aligned}
S_1 &= \{1\} \\
S_2 &= \{1, 2\} \\
v(S_1) &= y_3 = z_1 + z_2 + z_3 \\
&= \tau_1 \left((1-\epsilon_1)\epsilon_2 + (1-\epsilon_2)\epsilon_1 + (1-\epsilon_1)(1-\epsilon_2)\right) \\
&= \tau_1(1-\epsilon_1\epsilon_2) \\
v(S_2) &= y_2 + y_5 = z_2 + z_3 + z_5 + z_6 \\
&= \tau_1 \left((1-\epsilon_2)\epsilon_1 + (1-\epsilon_1)(1-\epsilon_2)\right) + \tau_2 \left((1-\epsilon_4)\epsilon_3 + (1-\epsilon_3)(1-\epsilon_4)\right) \\
&= \tau_1(1-\epsilon_2) + \tau_2(1-\epsilon_4)
\end{aligned}
$$

n) State the min-cut capacity r for a flow from s to t in dependency of τ_1 and τ_2.

$$
r = \min\left\{v(S_1), v_(S_2)\right\} = \min\left\{\tau_1(1-\epsilon_1\epsilon_2), \tau_1(1-\epsilon_2) + \tau_2(1-\epsilon_4)\right\}
$$

o) Determine τ_1 and τ_2 such that r is maximized.

We need to solve the optimization problem

$$
r^* = \max_{\substack{\tau_1,\tau_2 \geq 0 \\ \tau_1+\tau_2=1}} \min\left\{\tau_1(1-\epsilon_1\epsilon_2), \tau_1(1-\epsilon_2) + \tau_2(1-\epsilon_4)\right\}.
$$

In case that $v(S_1) \neq v(S_2)$ we will increase the smaller one, which might decrease the larger one. The optimal solution is found when we either cannot further increase the value of the smaller cut or when $v(S_1) = v(S_2)$.

From the induced graph (see solution of (d)) we see that node 2 cannot contribute if $\epsilon_4 > \epsilon_2$. In this case only node 1 will transmit and thaus $\tau_1 = 1$ and $\tau_2 = 0$. The same is obviously true when $\epsilon_1 = 1$ since node 2 cannot receive anything from node 1 in this case.
For $\epsilon_4 \leq \epsilon_2$, $\epsilon_1 < 1$, and $\tau_1 = 1$ we find that $v(S_1) > v(S_2)$. We therefore increase τ_2 at the cost of τ_1 until $v_(S_1) = v_(S_2)$, which is the optimal solution:

$$
\begin{aligned}
\tau_1 + \tau_2 = 1 \quad &\Rightarrow \quad \tau_2 = 1 - \tau_1 \\
v(S_1) &= \tau_1(1-\epsilon_1\epsilon_2) \\
v(S_2) &= \tau_1(1-\epsilon_2) + \tau_2(1-\epsilon_4) \\
&= \tau_1(\epsilon_4 - \epsilon_2) + 1 - \epsilon_4 \\
v(S_1) &\overset{!}{=} v(S_2) \\
\tau_1(1-\epsilon_1\epsilon_2) &= \tau_1(\epsilon_4 - \epsilon_2) + 1 - \epsilon_4 \\
\tau_1(1 - \epsilon_4 - \epsilon_1\epsilon_2 + \epsilon_2) &= 1 - \epsilon_4 \\
\tau_1 &= \frac{1-\epsilon_4}{1-\epsilon_4-\epsilon_1\epsilon_2+\epsilon_2}
\end{aligned}
$$

We therefore get the following solution:

$$\tau_1 = \begin{cases} 1 & \epsilon_1 = 1 \vee \epsilon_2 \leq \epsilon_4, \\ \frac{1-\epsilon_4}{1-\epsilon_4-\epsilon_1\epsilon_2+\epsilon_2} & \epsilon_2 > \epsilon_4. \end{cases}$$

Note that we could modify the cases such that $\epsilon_2 < \epsilon_4$ and $\epsilon_2 \geq \epsilon_4$ without affecting the capacity.

We now consider the multicast $s = 1$ and $T = \{2, 3\}$.

p) Determine the missing $s - T$ cut and its capacity.

$S_3 = \{1, 3\}$ with

$$v(S_3) = y_1 + y_7 = z_1 + z_3 + z_7 = \tau_1(1 - \epsilon_1) + \tau_3(1 - \epsilon_5)$$

q) State the optimization problem to maximize the multicast capacity r'.

$$\max_{\substack{\boldsymbol{\tau} \geq \mathbf{0} \\ \mathbf{1}^T \boldsymbol{\tau} = 1}} \min \{v(S_1), v(S_2), v(S_3)\}$$

r) Determine the maximum multicast rate r'^* by solving the problem.

Hint: It is sufficient to differentiate between cases and to express τ_2, τ_3 by means of τ_1. Except for the trivial case, the expression for τ_1 is not nice.

$$\begin{aligned} \tau_1 + \tau_2 + \tau_3 &= 1 \\ v(S_1) &= \tau_1(1 - \epsilon_1\epsilon_2) \\ v(S_2) &= \tau_1(1 - \epsilon_2) + \tau_2(1 - \epsilon_4) \\ v(S_3) &= \tau_1(1 - \epsilon_1) + \tau_3(1 - \epsilon_5) \end{aligned}$$

From the solution of (d) we can derive the following four cases:

1. $\epsilon_2 \leq \epsilon_4 \ \wedge \ \epsilon_1 \leq \epsilon_5$:
   In this case neither node 3 can help relaying data to node 2 nor node 2 can help relaying data to node 3 since in any case the arcs originating at 1 have the lowest erasure probabilities. Consequently we have that $\tau_1 = 1$ and $\tau_2 = \tau_3 = 0$.

2. $\epsilon_2 \leq \epsilon_4 \ \wedge \ \epsilon_1 > \epsilon_5$:
   Node 2 is still unable to help realying but node 3 now has a better link to node 2. Therefore, we have that $\tau_1, \tau_3 > 0$ and $\tau_3 = 0$. This gives the following set of equations:

$$\begin{aligned} \tau_1 + \tau_3 = 1 \ &\Rightarrow \ \tau_3 = 1 - \tau_1 \\ v(S_1) &= \tau_1(1 - \epsilon_1\epsilon_2) \\ v(S_2) &= \tau_1(1 - \epsilon_2) \\ v(S_3) &= \tau_1(1 - \epsilon_1) + \tau_3(1 - \epsilon_5) \end{aligned}$$

We now see that $v(S_1) \geq v(S_2)$. We therefore set $v(S_2) = v(S_3)$ which gives the opimal solution:

$$\begin{aligned}
\tau_1 &= \frac{1-\epsilon_5}{1-\epsilon_2-\epsilon_5+\epsilon_1}, \\
\tau_2 &= 0, \\
\tau_3 &= 1-\tau_1.
\end{aligned}$$

3. $\epsilon_2 > \epsilon_4 \;\wedge\; \epsilon_1 \leq \epsilon_5$: This case is similar to the previous: node 2 can now help relaying messages to node 3 but node 3 is unable to help relaying to node 2. Consequently we have that $\tau_3 = 0$ and $\tau_1, \tau_2 > 0$, which gives the following equations:

$$\begin{aligned}
\tau_1 + \tau_2 = 1 \;\Rightarrow\; \tau_2 &= 1 - \tau_2 \\
v(S_1) &= \tau_1(1-\epsilon_1\epsilon_2) \\
v(S_2) &= \tau_1(1-\epsilon_2) + \tau_2(1-\epsilon_4) \\
v(S_3) &= \tau_1(1-\epsilon_1)
\end{aligned}$$

Now we se that $v(S_1) \geq v_(S_3)$. Therefore, we again set $v(S_2) = v(S_3)$ and obtain:

$$\begin{aligned}
\tau_1 &= \frac{1-\epsilon_4}{1-\epsilon_1-\epsilon_4+\epsilon_2}, \\
\tau_2 &= 1-\tau_1, \\
\tau_3 &= 0.
\end{aligned}$$

4. $\epsilon_2 > \epsilon_4 \;\wedge\; \epsilon_1 > \epsilon_5$: Now both nodes 2 and 3 can help relaying messages to each other. Therefore, we have that $\tau_1, \tau_2, \tau_3 > 0$:

$$\begin{aligned}
\tau_1 + \tau_2 + \tau_3 = 1& \\
v(S_1) &= \tau_1(1-\epsilon_1\epsilon_2) \\
v(S_2) &= \tau_1(1-\epsilon_2) + \tau_2(1-\epsilon_4) \\
v(S_3) &= \tau_1(1-\epsilon_1) + \tau_3(1-\epsilon_5)
\end{aligned}$$

We now set $v(S_1) = v(S_2) = v(S_3)$, i.e., all three cuts are binding, and express τ_2 and τ_3 by means of τ_1:

$$\begin{aligned}
\tau_2 &= \tau_1 \frac{\epsilon_2(1-\epsilon_1)}{1-\epsilon_4}, \\
\tau_3 &= \tau_1 \frac{\epsilon_1(1-\epsilon_2}{1-\epsilon_5}.
\end{aligned}$$

Using $\tau_1 + \tau_2 + \tau_3 = 1$ we finally obtain

$$\begin{aligned}
\tau_1 &= \frac{(1-\epsilon_4)(1-\epsilon_5)}{\epsilon}, \\
\tau_2 &= \frac{\epsilon_2(1-\epsilon_1)(1-\epsilon_5)}{\epsilon}, \\
\tau_3 &= \frac{\epsilon_1(1-\epsilon_2)(1-\epsilon_4)}{\epsilon},
\end{aligned}$$

with $\epsilon = (1-\epsilon_4)(1-\epsilon_5) + \epsilon_1(1-\epsilon_2)(1-\epsilon_4) + \epsilon_2(1-\epsilon_1)(1-\epsilon_5)$.