**Chair for Network Architectures and Services**
Department of Informatics
TU München – Prof. Carle

# Network Security

## Chapter 2 – Basics
## 2.2 Public Key Cryptography

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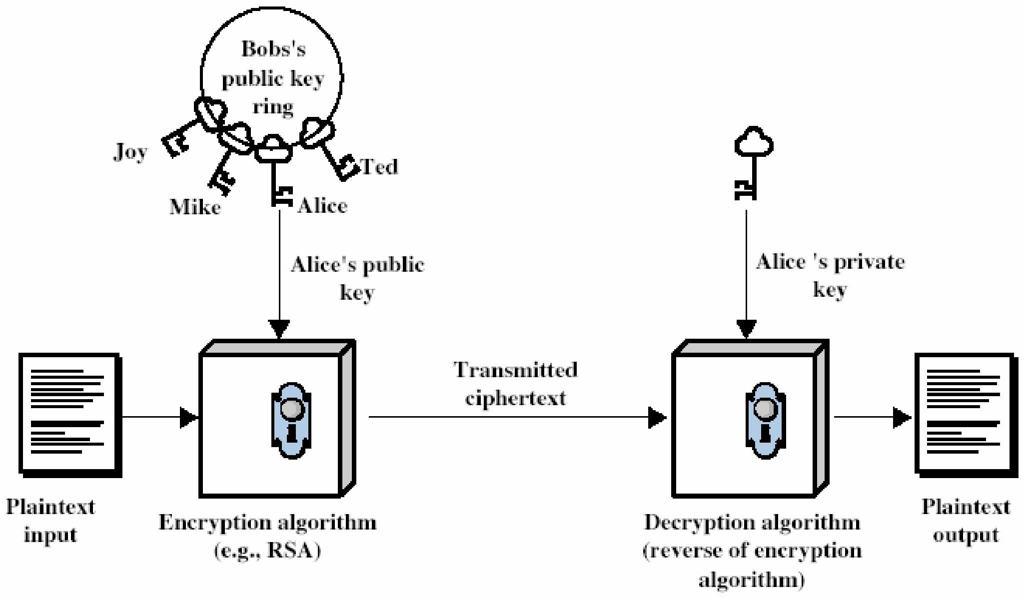
## Encryption/Decryption using Public Key Cryptography



**General Idea: encrypt with a publicly known key, but decryption only possible with a secret = private key**

# Public Key Cryptography

- General idea:
  - Use two different keys
    - a private key $K_{priv}$
    - a public key $K_{pub}$
  - Given a ciphertext $c = E(K_{pub}, m)$ and $K_{pub}$ it should be *infeasible* to compute the corresponding plaintext without the private key $K_{priv}$:

    $m = D(K_{priv}, c) = D(K_{priv}, E(K_{pub}, m))$
  - It must also be infeasible to compute $K_{priv}$ when given $K_{pub}$
  - The key $K_{priv}$ is only known to the owner entity A

    → called A's *private key* $K_{priv\text{-}A}$
  - The key $K_{pub}$ can be publicly known and is called A's *public key* $K_{pub\text{-}A}$

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# Public Key Cryptography

- Applications:
  - Encryption: If B encrypts a message with A's public key $K_{pub\text{-}A}$, he can be sure that only A can decrypt it using $K_{priv\text{-}A}$
  - Signing: digital signatures

- Important:
  - If B wants to communicate with A, he needs to verify that he really knows A's public key and does not accidentally use the key of an adversary
  - Known as the "*binding of a key to an identity*"
  - Not a trivial problem – so-called Public Key Infrastructures are one "solution"
    - X.509
    - GnuPG Web of Trust

# Public Key Cryptography

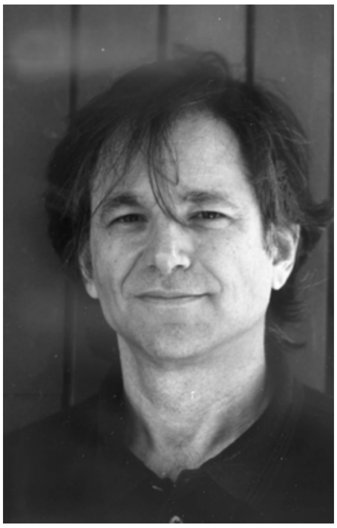
- Ingredients for a public key crypto system:
  - One-way functions: It is believed that there are certain functions that are easy compute, while the inverse function is very *hard* to compute
    - Real-world analogon: phone book
  - When we speak of *easy* and *hard*, we refer to certain complexity classes → more about that in crypto lectures and complexity theorey
  - For us: *Hard* means “infeasible on current hardware”
  - We know candidates, but have no proof for the existence of such functions
    - Existence would imply **P != NP**
- Special variant: Trap door functions
  - Same as one-way functions, but if a *second (“secret”) information* is known, then the inverse is *easy* as well
- Blueprint: use a trap-door function in your crypto system
- Candidates:
  - **Factorization problem**: basis of the RSA algorithm
    - Complexity class unknown, but assumed to be outside **P**
  - **Discrete logarithm problem**: basis of Diffie-Hellman and ElGamal
    - No polynomial algorithms known, assumed to be outside **P**

# The RSA Public Key Algorithm

- The RSA algorithm was described in 1977 by R. Rivest, A. Shamir and L. Adleman [RSA78]

Ron Rivest

Adi Shamir



Leonard Adleman

- Note: Clifford Cocks in the UK came up with the same scheme in 1973 – but he worked for the government and it was treated classified and thus remained unknown to the scientific community.

# Some Mathematical Background

- Definition: *Euler's Φ Function:*

  Let $\Phi(n)$ denote the number of positive integers $m < n$, such that $m$ is relatively prime to $n$.

  → "$m$ is relatively prime to $n$" = the greatest common divisor (gcd) of $m$ and $n$ is one.
- Let $p$ prime, then $\{1,2,\ldots,p-1\}$ are relatively prime to $p$, $\Rightarrow \Phi(p) = p-1$
- Let $p$ and $q$ distinct prime numbers and $n = p \times q$, *then*

  $\Phi(n) = (p-1) \times (q-1)$

- Euler's Theorem:

  Let $n$ and $a$ be positive and relatively prime integers,

  $\Rightarrow a^{\Phi(n)} \equiv 1 \text{ MOD } n$
  - Proof: see [Niv80a]

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# The RSA Public Key Algorithm

- RSA Key Generation:
  - Randomly choose $p$, $q$ distinct and large primes
    (really large: hundreds of bits = 100-200 digits each)
  - Compute $n = p \times q$, calculate $\Phi(n) = (p-1) \times (q-1)$ *(Euler's Φ Function)*
  - Pick $e \in Z$ such that $1 < e < \Phi(n)$ and $e$ is relatively prime to $\Phi(n)$, i.e. $\gcd(e,\Phi(n)) = 1$
  - Use the extended Euclidean algorithm to compute $d$ such that

    $e \times d \equiv 1 \text{ MOD } \Phi(n)$

  - The public key is $(n, e)$
  - The private key is $d$ – this is the "trap door information"

# The RSA Public Key Algorithm

- Definition: RSA function
  - Let $p$ and $q$ be large primes; let $n = p \times q$.
    Let $e \in N$ be relatively prime to $\Phi(n)$.
  - Then $\mathrm{RSA}(e,n) := x \rightarrow x^e \text{ MOD } n$
- Example:
  - Let $M$ be an integer that represents the message to be encrypted, with $M$ positive, smaller than $n$.
    - Example: Encode with <blank> = 99, A = 10, B = 11, ..., Z = 35
      So “HELLO” would be encoded as 1714212124.
      If necessary, break M into blocks of smaller messages: 17142   12124
  - To encrypt, compute: $C \equiv M^e \text{ MOD } n$
- Decryption:
  - To decrypt, compute: $M' \equiv C^d \text{ MOD } n$

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# The RSA Public Key Algorithm

- Why does RSA work:
  - As $d \times e \equiv 1 \text{ MOD } \Phi(n)$

  $\Rightarrow \exists\, k \in Z: \quad (d \times e) = 1 + k \times \Phi(n)$

  We sketch the “proof” for the case where M and n are relatively prime

$$
\begin{aligned}
M' &\equiv C^d \text{ MOD n} \\
&\equiv (M^e)^d \text{ MOD n} \\
&\equiv M^{(e \times d)} \text{ MOD n} \\
&\equiv M^{(1 + k \times \Phi(n))} \text{ MOD n} \\
&\equiv M \times (M^{\Phi(n)})^k \text{ MOD n} \\
&\equiv M \times 1^k \text{ MOD n} \qquad \text{(Euler's theorem*)} \\
&\equiv M \text{ MOD n} = \text{M}
\end{aligned}
$$

  - In case where M and n are not relatively prime, Euler’s theorem can not be applied.
  - See [Niv80a] for the complete proof in that case.

# Using RSA

- All public-key crypto systems are much slower and more resource-consuming than symmetric cryptography
- Thus, RSA is usually used in a hybrid way:
  - Encrypt the actual message with symmetric cryptography
  - Encrypt the symmetric key with RSA
- Using RSA requires some precautions
  - Careful with choosing $p$ and $q$: there are factorization algorithms for certain values that are very efficient
  - Generally, one also needs a *padding scheme* to prevent certain types of attacks against RSA
  - E.g. attack via Chinese remainder theorem: if the same clear text message is sent to $e$ or more recipients in an encrypted way, and the receivers share the same exponent $e$, it is easy to decrypt the original clear text message
  - Padding also works against a Meet-in-the-middle attack
  - OAEP (from PKCS#1) is a well-known padding scheme for RSA

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# On the Security of RSA

- The security of the RSA algorithm lies in the presumed difficulty of factoring $n = p \times q$
- It is known that computing the private key from the public key is as difficult as the factorization
- It is unknown if the private key is really needed for efficient decryption (there might be a way without, only no-one knows it yet)
- RSA is one of the most widely used – and studied – algorithms
- We need to increase key length regularly, as computers become more powerful
  - 633 bit keys have already been factored
  - Some claim 1024 bits may break in the near future (others disagree)
  - Current recommendation is 2048 bit, should be on the safe side
  - More is better, but slower

# Alternatives to RSA

- ElGamal (by Tahar El Gamal)

- Can be used for encryption and digital signatures
- ElGamal is based on another important “difficult” computational problem: Discrete logarithm (DLog)
- We discuss DLog soon
- We don’t discuss ElGamal in detail here, but it has practical relevance:
  - ElGamal is a default in GnuPG
  - Digital Signature Algorithm (DSA) is based on ElGamal
  - As such, ElGamal/DSA is also part of Digital Signature Standard (another NIST standard)
  - It is mathematically interesting because it adds a random component to encryption

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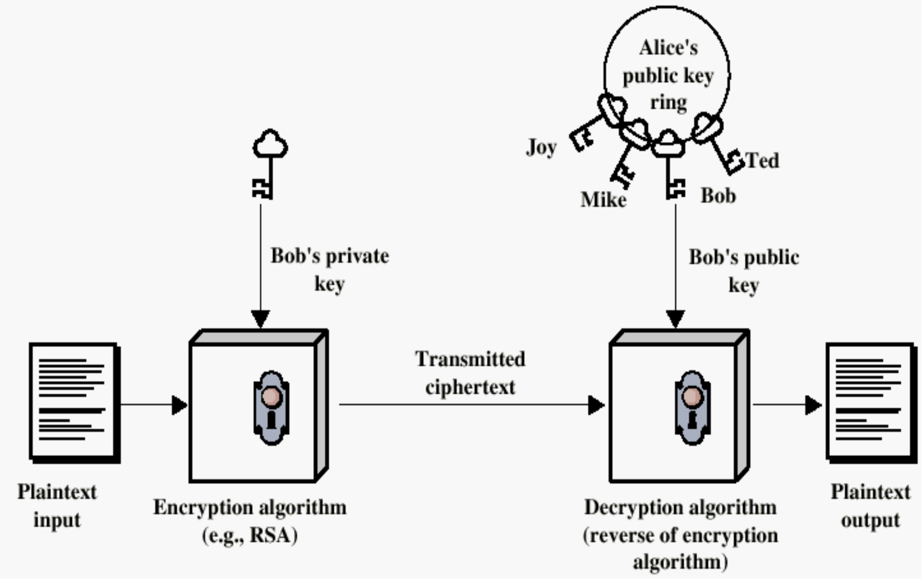
# Digital Signatures



- Signing = adding a proof of who has created a message, and that it has not been altered on the way
  - Who: authenticity
  - Not altered: integrity

# Digital Signatures

- A wants to sign a message. General idea:
  - A computes a cryptographic *hash value* of her message: h(m)
    - Hashes are one-way functions, i.e. given h(m) it's infeasible to obtain m
    - We'll discuss hash functions soon
  - A encrypts h(m) with her *private* key $K_{priv\text{-}A}$ →_Sig = $E_{K\_priv}(h(m))$
  - Given m, everyone can now
    - compute h(m)
    - Decrypt signature: D(E(h(m))) = h(m) and check if hash values are the same
  - If they match, A must have been the creator as only A knows the private key

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# Digital Signatures in Practice

- RSA
  - As $(d \times e) = (e \times d)$ , the operation also works in the opposite direction, i.e. it is possible to encrypt with *d* and decrypt with *e*
  - This property allows to use the two keys *d* and *e* for encryption and signatures
- DSA: signature method based on ElGamal/Dlog

- Important: sign message first or encrypt first?
  - Wrong: sign encrypted data only: with *c = E(m)*, send *c,Sig(c)*
    - Attacker can just strip signature and replace it with his own – and receiver cannot determine *who* has sent the message
    - Correct way: never sign ciphertexts – sign the message and send *c,Sig(m)*
  - Wrong: send *E(m,Sig(m))* without including destination
    - "Surreptitious forwarding" becomes possible: receiver B can decrypt, re-encrypt and replace receiver with some entity C and claim message was always for C
    - Correct way: always include receiver in signature: *E(B,m,Sig(B,m))*
  - Thus, use it correctly
- With current weaknesses in hash algorithms (MD5, SHA1), sending *E(B,m,Sig(B,m))* may currently be more secure

# The Discrete Logarithm: DLog

- In the following, we will discuss another popular one-way / trap-door function: the discrete logarithm
- DLog is used in a number of ways
  - Diffie-Hellman Key Agreement Protocol
    - “Can I agree on a key with someone else if the attacker can read my messages?”
  - ElGamal
  - DLog problems can be transformed to Elliptic Curve Cryptography
    - We’ll discuss this later
- Now: more mathematics

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# Some Mathematical Background

- Theorem/Definition: *primitive root, generator*
  - Let $p$ be prime. Then $\exists$ g $\in$ {1,2,…,p-1} such that

    $\{g^a \mid 1 \leq a \leq (p-1)\} = \{1,2,\ldots,p-1\}$ if everything is computed MOD p

  i.e. by exponentiating $g$ you can obtain all numbers between 1 and ($p$ -1)
  - For the proof see [Niv80a]
  - $g$ is called a primitive root (or generator) of {1,2,…,p-1}
- Example: Let $p$ = 7. Then 3 is a primitive root of {1,2,…,p-1}

  $1 \equiv 3^6$ MOD 7, $2 \equiv 3^2$ MOD 7, $3 \equiv 3^1$ MOD 7, $4 \equiv 3^4$ MOD 7,

  $5 \equiv 3^5$ MOD 7, $6 \equiv 3^3$ MOD 7

# DLog: Some Mathematical Background

- Definition: discrete logarithm
  - Let *p* be prime, *g* be a primitive root of {1,2,…,p-1} and *c* be any element of {1,2,…,p-1}. Then ∃ *z* such that: $g^z \equiv c$ MOD *p*

    *z* is called the discrete logarithm of *c* modulo *p* to the base *g*
  - Example: 6 is the discrete logarithm of 1 modulo 7 to the base 3 as 36 ≡ 1 MOD 7
  - The calculation of the discrete logarithm *z* when given *g*, *c*, and *p* is a computationally difficult problem and the asymptotical runtime of the best known algorithms for this problem is exponential in the bit-length of p

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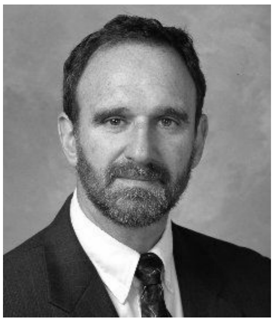
# Diffie-Hellman Key Exchange (1)

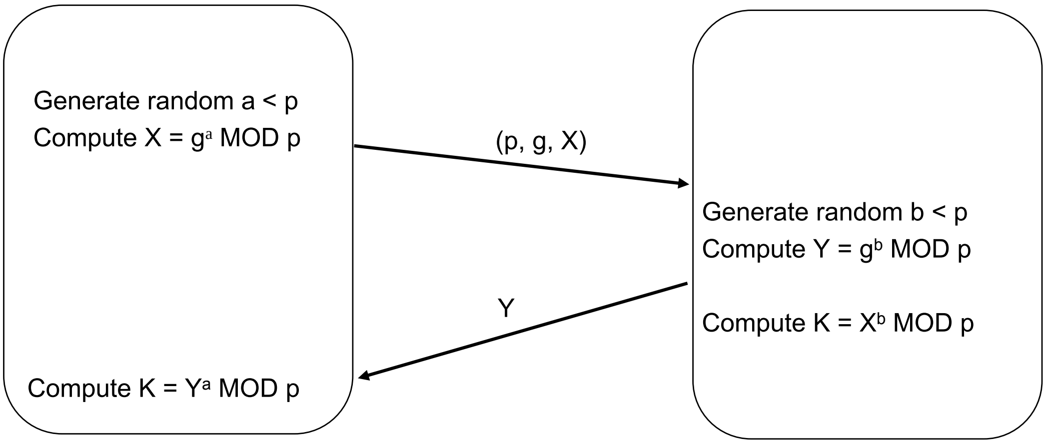
- The Diffie-Hellman key exchange was first published in the landmark paper [DH76], which also introduced the fundamental idea of asymmetric cryptography
- The DH exchange in its basic form enables two parties A and B to agree upon a *shared secret* using a public channel:
  - *Public channel* means, that a potential attacker can read all messages exchanged between A and B
  - It is important that A and B can be sure that the attacker is not able to alter messages as in this case he might launch a *man-in-the-middle attack*
  - The mathematical basis for the DH exchange is the problem of finding *discrete logarithms in finite fields*
  - The DH exchange is *not* an encryption algorithm.

# Diffie-Hellman Key Exchange (2)

Whitfield Diffie



Martin E. Hellman



Generate random $a < p$
Compute $X = g^a \text{ MOD } p$

$(p, g, X)$

Generate random $b < p$
Compute $Y = g^b \text{ MOD } p$

Compute $K = X^b \text{ MOD } p$

$Y$

Compute $K = Y^a \text{ MOD } p$

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# Diffie-Hellman Key Exchange (3)

- If Alice (*A*) and Bob (*B*) want to agree on a shared secret *K* and their only means of communication is a public channel, they can proceed as follows:
- *A* chooses a prime *p,* a primitive root *g* of {1,2,…,p-1} and a random number *x*
- *A* and *B* can agree upon the values *p* and *g* prior to any communication, or *A* can choose *p* and *g* and send them with his first message
- *A* chooses a random number *a:*
- *A* computes $X = g^a \text{ MOD } p$ and sends *X* to *B*
- B chooses a random number *b*
- *B* computes $Y = g^b \text{ MOD } p$ and sends *Y* to *A*
- Both sides compute the common secret:
  - *A* computes $K = Y^a \text{ MOD } p$
  - *B* computes $K' = X^b \text{ MOD } p$
  - As $g^{(a \cdot b)} \text{ MOD } p = g^{(b \cdot a)} \text{ MOD } p$, it holds: $K = K'$
- An attacker Eve who is listening to the public channel can only compute the secret *K*, if she is able to compute either *a* or *b* which are the discrete logarithms of *X* and *Y* modulo *p* to the base *g.*
- In essence, A and B have agreed on a key *without ever sending the key over the channel*
- This does not work anymore if an attacker is on the channel and can replace the values with his own ones

# Elliptic Curve Cryptography (ECC)

- Motivation: RSA is probably the most widely implemented algorithm for Public Key Cryptography
  - Does public key cryptography need long keys with 1024-8192 bits?
  - Also, it is good to think of alternatives due to the developments in the area of primality testing, factorization and computation of discrete logarithms

  → Elliptic Curve Cryptocraphy (ECC)
- ECC is based on a finite field of points.
- Points are presented within a 2-dimensional coordinate system: $(x,y)$
- All points within the elliptic curve satisfy an equation of this type:

$$y^2 = x^3 + ax + b$$

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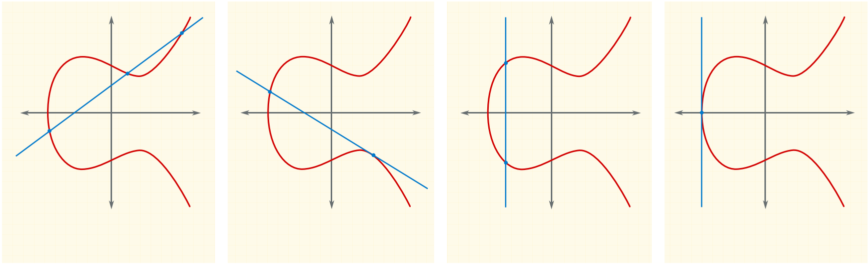
# Elliptic Curve Cryptography (ECC)

- Given this set of points an additive operator can be defined



- A multiplication of a point P by a number n is simply the addition of P to itself n times

$$Q = nP = P + P + \ldots + P$$

- The problem of determining n, given P and Q, is called the elliptic curve's discrete logarithm problem (ECDLP)

1

*R*

- The ECDLP is believed to be hard in the general class obtained from the group of points on an elliptic curve over a finite field

*Q*

# Elliptic Curve Cryptography (ECC)

- Any DLog-based algorithm can be turned into an ECC-based algorithm
- ECC problems are generally believed to be "harder"
  (though there is a lack of mathematic proofs)
- Allows us to have shorter key sizes
  → good for storage and transmission over networks
- ECC is still "a new thing" → but there are more implementations now

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# Key Length (1)

- It is difficult to give good recommendations for appropriate and secure key lengths
- Hardware is getting faster
- So key lengths that might be considered as secure this year, might become insecure in 2 years
- Adi Shamir published in 2003 [Sham03] a concept for breaking 1024 bits RSA key with a special hardware within a year (hardware costs were estimated at 10 Millions US Dollars)
- Bruce Schneier recommends in [Fer03] a minimal length of 2048 bits for RSA "if you want to protect your data for 20 years"
- He recommends also the use of 4096 and up to 8192 bits RSA keys

# Key Length (2)

- Comparison of the security of different cryptographic algorithms with different key lengths
  - Note: this is an informal way of comparing the complexity of breaking an encryption algorithm
  - So please be careful when using this table
  - Note also: a symmetric algorithm is supposed to have no significant better attack that breaks it than a brute-force attack

| Symmetric | RSA | ECC |
|---|---|---|
| 56 | 622 | 105 |
| 64 | 777 | 120 |
| 74 | 1024 | 139 |
| 103 | 2054 | 194 |
| 128 | 3214 | 256 |
| 192 | 7680 | 384 |
| 256 | 15360 | 512 |

Source [Bless05] page 89

# Summary

- Public key cryptography allows to use two different keys for:
  - Encryption / Decryption
  - Digital Signing / Verifying
- Some practical algorithms that are still considered to be secure:
  - RSA, based on the difficulty of factoring
  - Diffie-Hellman (a key agreement protocol)
- As their security is entirely based on the difficulty of certain number theory problems, algorithmic advances constitute their biggest threat
- Practical considerations:
  - Public key cryptographic operations are magnitudes slower than symmetric ones
  - Public cryptography is often just used to exchange a symmetric *session key* securely, which is on turn will be used for to secure the data itself.

# Additional References

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